IE170: Algorithms in Systems Engineering: Lecture 7

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Taking Stock

Last Time

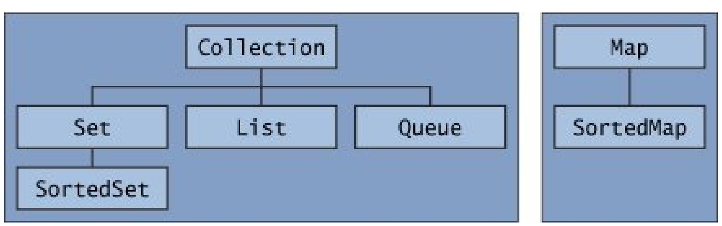
- Master Theorem Practice
- Some Sorting Algs
- Beginning of Java Collections Interfaces

This Time

- Hashes
- Trees and Binary Search Trees
- More on Java Collections Interfaces
- `lab == fun`

The Java Collections Interfaces

- In the remainder of the class, we will be using the Java Collections Interface: `http://java.sun.com/docs/books/tutorial/collections/TOC.html`
- Important: Most of what I will say only works if you set the "code level" to Java 5.0 in eclipse!
- **Preferences, Java Compiler**: Set this to ≥ 5.0
- The interfaces form a hierarchy:



(A subset of) the Collections Interface

```
public interface Collection<E> extends Iterable<E> {
    // Basic operations
    int size();
    boolean isEmpty();
    boolean contains(Object element);
    boolean add(E element);         //optional
    boolean remove(Object element); //optional
    Iterator<E> iterator();

    // Array operations
    Object[] toArray();
    <T> T[] toArray(T[] a);
}
```

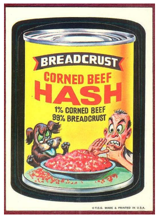
Set

- A Set is a Collection that cannot contain duplicate elements.
- It models the mathematical set abstraction.
- The Set interface contains only methods inherited from Collection and adds the restriction that duplicate elements are prohibited.
- Set is still an interface. There are 3 implementations of Set in Java.
  - HashSet
  - TreeSet
  - LinkedHashSet

Hash?

- No, Cheech. A hash table is a data structure in which we can "look up" (or search) for an element efficiently.
- The expected time to search for an element in a has table in $O(1)$. (Worst case time in $\Theta(n)$).
- Think of a hash table as an array
- With a regular array, we find the element whose "key" is j in position j of the array. `j = 17; val = a[j];` .
- This is called *direct addressing* and it takes $O(1)$ on your regular ol' random access computer.
- This form of direct addressing works when we can afford to have an array with one position for every possible key

More on Hash



- In a hash table the number of keys stored is small relative to the number of possible keys
- A hash table is an array. Given a key k, we don't use k as the index into the array – rather, we have a hash function h, and we use $h(k)$ as an index into the array.
- Given a "universe" of keys K.
  - Think of K as all the words in a dictionary, for example
- $h : K \rightarrow \{0, 1, \ldots m - 1\}$, so that $h(k)$ gets mapped to an integer between 0 and $m - 1$ for every $k \in K$
- We say that k hashes to $h(k)$

Example

- This look great. However, what happens if $h(k_1) = h(k_2)$ for $k_1 \neq k_2$?
- Two keys hash to the same value. The elements collide
- This is typically handled by chaining
- Instead of storing a key k (or later key value pair (k, v)) at every position in the array, we store a linked list of keys.
- Example:

A (Fairly) Obvious point

- BAD hash function. $h(k) = 3$.
- If all keys hash to the same value, then looking up a key takes $\Theta(n)$. (Since it is just a list).
- We would like a hash function to be "random" in the sense that a key k is equally likely to has into any of the m slots in the hash table (array).
- If we have such a function, then we can show that the average time required to search for a key is $\Theta(1 + \frac{n}{m})$
- When hashing keys that are not numbers, you must convert them to numbers, e.g.:

$$\text{BEER} = -142 + 2^4 + 5^3 + 5^2 + 18^1 = 42.$$

Average Hash Search Time

- The number of elements to be searched is 1 more than the number of elements that appear before x in $x's$ list. Assuming we insert items into the list at the beginning, then this is the number of elements that were inserted after x.
- By definition: $\mathbb{P}(h(k_i) = h(k_j)) = \frac{1}{m}$
- Let X_{ij} be indicator random variable that is equal to one if and only if $h(k_i) = h(k_j)$
- Then just compute:

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(1 + \sum_{j=i+1}^{n} X_{ij}\right)\right].$$

Hash Functions

Modular Hash Function

- Let m be (roughly) the size of your hash table:
- $h(k) = k \bmod m$
- Good choice of m: A prime number not too close to an exact power of 2

Multiplicative Hash Function

- $h(k) = \lfloor m(kA \bmod 1) \rfloor$
- Multiply key k by A, take fractional part, and multiply by m
- If $m = 2^p$ this can be done very fast with bit shifting
- $A \approx \phi = (\sqrt{5} - 1)/2$ seems a good value

Back to the Java Collections

- So now you know what a Java `HashSet` is.
- A `LinkedHashSet` is a `HashSet` that also keeps track of the order in which elements were inserted.
- (Think of laying a linked list on top of the Hash Table)
- A `TreeSet` stores its elements in a alertred-black tree.
- In order to understand red-black trees, we must know about binary search trees.
- Hash table is "good" at INSERT(), SEARCH(), DELETE(). But what if you also want to support (efficiently) MINIMUM(), MAXIMUM()

Trees

- A tree is a set of items organized into a hierarchical structure (think of a family tree).
- When organized in this way, we call the items nodes.
- Each node has a single designated parent and one or more children.
- There is a single designated node, called the root, with no parent.
- Any node with no children is called a leaf.
- Any node with children is called internal.
- A tree in which all nodes have 2 or fewer children is called a binary tree.
- Storing a list of items in a tree structure allows us to represent additional relationships among the items in the list.

Binary Tree Data Structures

- To store a tree of keys k, or maybe (key, value) pairs: (k, v), we need a data structure supporting three basic operations
  - left l: Points to the left child
  - right r: Points to the right child
  - parent p: Points to the parent
- This allows us to traverse the tree and perform other operations on it.
- The level of a node in the tree is the number of recursive calls to `parent()` needed to reach the root.
- The depth of the tree is the maximum level of any of its nodes.
- A balanced tree is one in which all leaves are at levels k or $k-1$, where k is the depth of the tree.

Data Structures for Storing Trees

Array

- The root is stored in position 0.
- The children of the node in position i are stored in positions $2i+1$ and $2i+2$.
- This determines a unique storage location for every node in the tree and makes it easy to find a node's parent and children.
- Using an array, the basic operations can be performed very efficiently.
- If the tree is unbalanced or dynamic, a linked list may be better.

Data Structures for Storing Trees

Linked List

- In a linked list, each item is stored along with explicit pointers to its parent and children.
- This allows for easy addition and deletion of nodes from the tree.

Binary Search Tree

- A binary search tree is a data structrue that is conceptualized as a binary tree, but has one additional property:

Binary Search Tree Property

If y is in the left subtree of x, then $k(y) \leq k(x)$

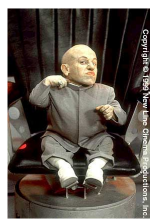
Binary Search Trees

- There are lots of binary trees that can satisfy this property.
- It is obvious that the number of binary tree on n nodes b_n is

$$b_n = \frac{1}{n+1}\binom{2n}{n} \qquad b_n = \frac{4^n}{\sqrt{\pi}n^{3/2}}(1 + O(1/n))$$

- And not all of these (exponentially many) are created equal.
- In fact, we would like to keep our binary search trees "short", because most of the operations we would like to support are a function of the height h of the tree.

Short Is Beautiful



- $\textsc{Search}()$ takes $O(h)$
- $\textsc{Minimum}()$, $\textsc{Maximum}()$ also take $O(h)$
- Slightly less obvious is that $\textsc{Insert}()$, $\textsc{Delete}()$ also take $O(h)$
- Thus we would like to keep out binary search trees "short" (h is small).

Operations

$\textsc{Successor}(x)$

- How would I know "next biggest" element?
- If right subtree is not empty: $\textsc{Minimum}(r(x))$
- If right subtree is empty: Walk up tree until you make the first "right" move

$\textsc{Insert}(x)$

- Just walk down the tree and put it in. It will go "at the bottom"

DELETE()

- If 0 or 1 child, deletion is fairly easy
- If 2 children, deletion is made easier by the following fact:

Binary Search Tree Property

- If a node has 2 children, then
  - its successor will not have a left child
  - its predecessor will not have a right child

red-black Trees

- red-black trees are simply a way to keep binary search trees short. (Or balanced)
- Balanced here means that no path on the tree is more than twice as long as another path.
- An implication of this is that its maximum height is $2\lg(n+1)$
- SEARCH(), MINIMUM(), MAXIMUM(), all take $O(\lg n)$
- It's implementation is complicated, so we won't cover it
- INSERT(): also runs in $O\lg(n)$
- DELETE(): runs in $O\lg(n)$
  - (but it is more complicated to maintain the "red-black" property)

Back to the Java Collections

- red-black trees remain sorted
- You don't really have any control over the order in which things will appear in a `HashSet`
- If you care about that – you should use a `LinkedHashSet`, which lays a linked list on top of the HashSet
- In general, Sets are not for ordered collections of items, for that, you should use a list

Lists

- A List is an ordered Collection (sometimes called a sequence).
- Lists may contain duplicate elements.
- In addition to the operations inherited from Collection, the List interface includes operations for the following:
  - Positional access: manipulate elements based on their numerical position in the list
  - Search: searches for a specified object in the list and returns its numerical position

(Subset of) List Interface

```
public interface List<E> extends Collection<E> {
    // Positional access
    E get(int index);
    E set(int index, E element);    //optional
    boolean add(E element);         //optional
    void add(int index, E element); //optional
    E remove(int index);            //optional

    // Search
    int indexOf(Object o);
    int lastIndexOf(Object o);

    // Iteration
    ListIterator<E> listIterator();
    ListIterator<E> listIterator(int index);

}
```


Java List Implementations

Two List Implementations

1. ArrayList: which is usually the better-performing
2. LinkedList: offers better performance under certain circumstances, (i.e. if lots of add/remove in the middle if the list)

Java Lists have extended iterators

```
public interface ListIterator<E> extends Iterator<E> {
    boolean hasNext();
    E next();
    boolean hasPrevious();
    E previous();
    int nextIndex();
    int previousIndex();
    void remove(); //optional
    void set(E e); //optional
    void add(E e); //optional
}
```


List Stuff

- `ListIterator<E> listIterator()`: gives iterator at beginning
- `ListIterator<E> listIterator(int index)`: gives iterator at specified index
- The index refers to the element that would be returned by an initial call to `next()`
- The cursor is always between two elements:
  - the one that would be returned by a call to `previous()`
  - the one that would be returned by a call to `next()`
- The $n+1$ valid index values correspond to the $n+1$ gaps between elements, from the gap before the first element to the gap after the last one.

Next Time

- A bit on Java Collection `Map` Interface
- Move on to Heaps (Chapter 6)
- We have covered chapters 1-4, 10-11, and Appendices A and B

News

- New Homework Posted!
- Let's have a LITTLE quiz on 2/7
- Homework is due 2/5: No late homework accepted. (I need to hand out solutions and discuss in class on 2/5).