IE418: Integer Programming

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Linear Algebra Review: Linear Independence

- A finite collection of vectors $x^1, \ldots, x^k \in \Re^n$ is *linearly independent* if the unique solution to $\sum_{i=1}^k \lambda_i x^i = 0$ is $\lambda_i = 0, \forall i = 1, 2, \ldots, k$. Otherwise, the vectors are *linearly dependent*.

Let A be a square matrix. Then, the following statements are equivalent:

- The matrix A is invertible.
- The matrix A^T is invertible.
- The determinant of A is nonzero.
- The rows of A are linearly independent.
- The columns of A are linearly independent.
- For every vector b, the system $Ax = b$ has a unique solution.

Linear Algebra Review: Affine Independence

- A finite collection of vectors $x^1, \ldots, x^k \in \Re^n$ is affinely independent if the unique solution to $\sum_{i=1}^k \alpha_i x^i = 0, \sum_{i=1}^k \alpha_i = 0$ is $\alpha_i = 0, \forall i = 1, 2, \ldots, k$.
- Linear independence implies affine independence, but not vice versa.
- Affine independence is essentially a "coordinate-free" version of linear independence.
- The following statements are equivalent:
  1. $x_1, \ldots, x_k \in \mathbb{R}^n$ are affinely independent.
  2. $x_2 - x_1, \ldots, x_k - x_1$ are linearly independent.
  3. $(x_1, 1), \ldots, (x_k, 1) \in \mathbb{R}^{n+1}$ are linearly independent.

Linear Algebra Review: Subspaces

- A nonempty subset $H \subseteq \mathbb{R}^n$ is called a subspace if $\alpha x + \gamma y \in H \; \forall x, y \in H$ and $\forall \alpha, \gamma \in \mathbb{R}$
- A linear combination of a collection of vectors $x^1, \ldots x^k \in \mathbb{R}^n$ is any vector $y \in \mathbb{R}^n$ such that $y = \sum_{i=1}^k \lambda_i x^i$ for some $\lambda \in \mathbb{R}^k$.
- The span of a collection of vectors $x^1, \ldots x^k \in \mathbb{R}^n$ is the set of all linear combinations of those vectors.
- Given a subspace $H \subseteq \mathbb{R}^n$, a collection of linearly independent vectors whose span is H is called a basis of H. The number of vectors in the basis is the dimension of the subspace.

Linear Algebra Review: Subspaces and Bases

- A given subspace has an infinite number of bases.
- Each basis has the same number of vectors in it.
- If S and T are subspaces such that $S \subseteq T \subseteq \mathbb{R}^n$, then a basis of S can be extended to a basis of T.
- The span of the columns of a matrix A is a subspace called the column space or the range, denoted $range(A)$.
- The span of the rows of a matrix A is a subspace called the row space.
- The dimensions of the column space and row space are always equal. We call this number $rank(A)$.

Linear Algebra Review: Rank and Nullity

- $rank(A) \leq \min\{m, n\}$. If $rank(A) = \min\{m, n\}$, then A is said to have full rank.
- The set $\{x \in \mathbb{R}^n \mid Ax = 0\}$ is called the nullspace of A (denoted $null(A)$) and has dimension $n - rank(A)$.

Some Properties of Subspaces

- The following are equivalent:
  1. $H \subseteq \mathbb{R}^n$ is a subspace.
  2. There is an $m \times n$ matrix A such that $H = \{x \in \mathbb{R}^n \mid Ax = 0\}$
  3. There is a $k \times n$ matrix B such that $H = \{x \in \mathbb{R}^n \mid x = uB, u \in \mathbb{R}^k\}$.
- If $\{x \in \mathbb{R}^n \mid Ax = b\} \neq \emptyset$, the maximum number of affinely independent solutions of $Ax = b$ is $n + 1 - rank(A)$.
- If $H \subseteq \mathbb{R}^n$ is a subspace, then $\{x \in \mathbb{R}^n \mid xy = 0 \text{ for } y \in H\}$ is a subspace called the orthogonal subspace and denoted $H^\perp$
- If $H = \{x \in \mathbb{R}^n \mid Ax = 0\}$, ($A \in \mathbb{R}^{m \times n}$) then $H^\perp = \{x \in \mathbb{R}^n \mid x = A^T u, u \in \mathbb{R}^m\}$

Convex Sets

- A set $S \subseteq \mathbb{R}^n$ is convex if $\forall x, y \in S, \lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda) y \in S$.
- Let $x^1, \ldots, x^k \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^k$ be given such that $\lambda^T e = 1$. Then
  1. The vector $\sum_{i=1}^k \lambda_i x^i$ is said to be a convex combination of $x^1, \ldots, x^k$
  2. The convex hull of $x^1, \ldots, x^k$ is the set of all convex combinations of these vectors, denoted $\text{conv}(x^1, \ldots, x^k)$.
- The convex hull of two points is a line segment.
- A set is convex if and only if for any two points in the set, the line segment joining those two points lies entirely in the set.
- All polyhedra are convex.

Polyhedra, Hyperplanes, and Half-spaces

- A polyhedron is a set of the form $\{x \in \mathbb{R}^n \mid Ax \leq b\} = \{x \in \mathbb{R}^n | a^i x \leq b^i, \ \forall i \in M\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.
- A polyhedron $P \subset \mathbb{R}^n$ is bounded if there exists a constant K such that $|x_i| < K \ \forall x \in P, \forall i \in [1, n]$.
- A bounded polyhedron is called a polytope.
- Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ be given.
  - The set $\{x \in \mathbb{R}^n \mid a^T x = b\}$ is called a hyperplane.
  - The set $\{x \in \mathbb{R}^n \mid a^T x \leq b\}$ is called a half-space.

Dimension of Polyhedra

- A polyhedron P is of dimension k, denoted $dim(P) = k$, if the maximum number of affinely independent points in P is $k + 1$.
- A polyhedron $P \subseteq \mathbb{R}^n$ is full-dimensional if $dim(P) = n$.
- Let
  - $M = \{1, \dots, m\}$,
  - $M^= = \{i \in M \mid a_i x = b_i \ \forall x \in P\}$ (the equality set,
  - $M^\leq = M \setminus M^=$ (the inequality set).
- Let $(A^=, b^=), (A^\leq, b^\leq)$ be the corresponding rows of (A, b).
- If $P \subseteq \mathbb{R}^n$, then $dim(P) + rank(A^=, b^=) = n$

Dimension and Rank

- $x \in P$ is called an inner point of P if $a^i x < b_i \ \forall i \in M^\leq$.
- $x \in P$ is called an interior point of P if $a^i x < b_i \ \forall i \in M$.
- Every nonempty polyhedron has an *inner point.*
- A polyhedron has an *interior point* if and only if it is *full-dimensional.*

- **2.4** If $P \subseteq \mathbb{R}^n$, then $dim(P) + rank(A^=, b^=) = n$

Example

- $POLLY \subseteq \mathbb{R}^5$:

$$
\begin{array}{rcr}
x_1 - 2x_2 + x_3 - x_4 + 2x_5 & \leq & 3 \\
x_1 - x_5 & \leq & 0 \\
-x_1 + x_5 & \leq & 0 \\
2x_2 - x_3 + x_4 & \leq & 2 \\
-4x_2 + 2x_3 - 2x_2 & \leq & -4 \\
3x_1 - x_2 & \leq & 2 \\
-x_1 & \leq & 0 \\
-x_2 & \leq & 0 \\
-x_3 & \leq & 0 \\
-x_4 & \leq & 0 \\
-x_5 & \leq & 0
\end{array}
$$

- What is $\dim(POLLY)$?
- Consider points:
  - $(1, 1, 0, 0, 1)$
  - $(0, 1, 0, 0, 0)$
  - $(1, 2, 2, 0, 1)$
  - $(0, 0, 0, 2, 0)$
  - $(1, 0, 0, 2, 1)$
- Which are in $POLLY$?

Figuring dim($POLLY$)

- Are the points in $POLLY$ affinely independent?

$$\text{rank}\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\right) = 4$$

- Implies that $(1, 1, 0, 0, 1)$, $(0, 1, 0, 0, 0)$, $(1, 2, 2, 0, 1)$, $(0, 0, 0, 2, 0)$ are affinely independent.
- dim($POLLY$) ≥ 3
- By **2.4**, we now know dim($POLLY$) $= 3, 4,$ or 5

Figuring dim($POLLY$)

- *All* points in $POLLY$ satisfy the following inequalities with equality:

$$\begin{aligned} x_1 - x_5 &\leq 0 \\ -x_1 + x_5 &\leq 0 \\ 2x_2 - x_3 + x_4 &\leq 2 \\ -4x_2 + 2x_3 - 2x_2 &\leq -4 \end{aligned}$$

$$\text{rank}\left(\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 2 \\ 0 & -4 & 2 & -2 & 0 & -4 \end{bmatrix}\right)$$

- So rank$(A^=, b^=) \geq 2$
- dim$(P) = 3$

Valid Inequalities and Faces

- The inequality denoted by (π, π_0) is called a valid inequality for P if $\pi x \leq \pi_0 \; \forall x \in P$.
- Note that (π, π_0) is a valid inequality if and only if P lies in the half-space $\{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\}$.
- If (π, π_0) is a valid inequality for P and $F = \{x \in P \mid \pi x = \pi_0\}$, F is called a face of P and we say that (π, π_0) represents or defines F.
- A face is said to be proper if $F \neq \emptyset$ and $F \neq P$.
- Note that a face has multiple representations.

More on Faces

- The face represented by (π, π_0) is nonempty if and only if max$\{\pi x \mid x \in P\} = \pi_0$.
- If the face F is nonempty, we say it supports P.
- Note that the set of optimal solutions to an LP is always a face of the feasible region.
- **3.1** Let P be a polyhedron with equality set $M^=$. If $F = \{x \in P | \pi^T x = \pi_0\}$ is nonempty, then F is a polyhedron. Let $M_F^= \supseteq M^=, M_F^\leq = M \setminus M_F^=$. Then $F = \{x \mid a_i^T x = b_i \forall i \in M_F^=, a_i^T x \leq b_i \forall i \in M_f^\leq\}$
  - We get the polyhedron F by taking some of the inequalities of P and making them equalities
  - The number of distinct faces of P is finite.

Facets

- A face F is said to be a facet of P if $dim(F) = dim(P) - 1$.
- In fact, facets are all we need to describe polyhedra.
- **3.2** If F is a facet of P, then in any description of P, there exists some inequality representing F. (By setting the inequality to equality, we get F).
- **3.3 and 3.4** Every inequality that represents a face that is not a facet is unnecessary in the description of P.

Example, cont.

Consider the face

$$F = \{x \in POLLY \mid 2x_1 + 10x_2 - 5x_3 + 5x_4 - 3x_5 = 10\}$$

- Is it proper?
  - $F = POLLY$?
  - $F = \emptyset$?
- Points: $(0,2,2,0,0), (0,1,0,0,0), (0,0,0,2,0)$
  - Are they in F? (Don't forget, they must also be in P)
  - Are they affinely independent?
  - **Yes!** so $\dim(F) \geq 2$
  - Is $\dim(F) \leq 2$? (Yes!)
  - $\dim(POLLY) = 3$, so F is a facet of $POLLY$

Facet Representation

- Remember **3.2**. If F is a facet of $POLLY$, then there is some inequality $a^T k \leq b_k, k \in M^{\leq}$ representing F.
- Which inequality in the inequality set of $POLLY$ represents F?

- $x \in POLLY, 2x_1 + 10x_2 - 5x_3 + 5x + 4 - 3x_5 = 10 \Rightarrow x_1 = 0$
  - So F is represented by the inequality $-x_1 \leq 0$

Polyhedra—A Fundamental Representation Theorem

Putting together what we have seen so far, we can say the following: (**3.5**)

- Every full-dimensional polyhedron P has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of P.
- If $dim(P) = n - k$ with $k > 0$, then P is described by a maximal set of linearly independent rows of $(A^=, b^=)$, as well as one inequality representing each facet of P.

Polyhedra—A Useful Facet Proving Theorem

- Put another way, if a facet F of P is represented by (π, π_0), then the set of all representations of F is obtained by taking scalar multiples of (π, π_0) plus linear combinations of the equality set of P.
- We can use this to actually prove an inequality is a facet! **(3.6)**

- Let $M^= \equiv (A^=, b^=)$ be the equality set of $P \subseteq \mathbb{R}^n$, and let $F = \{x \in P \mid \pi^T x = \pi_0\}$ be a proper face of P. The following statements are equivalent
  - F is a facet of P
  - If $\lambda x = \lambda_0 \ \forall x \in F$, then

$$(\lambda, \lambda_0) = (\alpha\pi + uA^=, \alpha\pi_0 + u^t b^=),$$

for some $\alpha \in \mathbb{R}, u \in \mathbb{R}^{|M^=|}$