IE418: Integer Programming

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Key Things We Learned Last Time

- If $\{x \in \mathbb{R}^n \mid Ax = b\} \neq \emptyset$, the maximum number of affinely independent solutions of $Ax = b$ is $n + 1 - rank(A)$.
- A polyhedron P is of dimension k, denoted $dim(P) = k$, if the maximum number of affinely independent points in P is $k + 1$.
- **2.4** If $P \subseteq \mathbb{R}^n$, then $dim(P) + rank(A^=, b^=) = n$
- Partition the inequalities defining P:
  - $M = \{1, \ldots, m\}$,
  - $M^= = \{i \in M \mid a_i x = b_i \ \forall x \in P\}$ (the equality set),
  - $M^{\leq} = M \setminus M^=$ (the inequality set).

Our Friend POLLY

$$
\begin{aligned}
x_1 - 2x_2 + x_3 - x_4 + 2x_5 &\leq 3 \\
x_1 - x_5 &\leq 0 \\
-x_1 + x_5 &\leq 0 \\
2x_2 - x_3 + x_4 &\leq 2 \\
-4x_2 + 2x_3 - 2x_4 &\leq -4 \\
3x_1 - x_2 &\leq 2 \\
-x_1 &\leq 0 \\
-x_2 &\leq 0 \\
-x_3 &\leq 0 \\
-x_4 &\leq 0 \\
-x_5 &\leq 0
\end{aligned}
$$

- $POLLY \subseteq \mathbb{R}^5$:
- Remember: $\dim(POLLY) = 3$

Valid Inequalities and Faces

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

- The inequality denoted by (π, π_0) is called a valid inequality for P if $\pi x \leq \pi_0 \ \forall x \in P$.
- Note that (π, π_0) is a valid inequality if and only if P lies in the half-space $\{x \in \mathbb{R}^n \mid \pi x \leq \pi_0\}$.
- If (π, π_0) is a valid inequality for P and $F = \{x \in P \mid \pi x = \pi_0\}$, F is called a face of P and we say that (π, π_0) represents or defines F.
- A face is said to be proper if $F \neq \emptyset$ and $F \neq P$.
- Note that a face has multiple representations.

More on Faces

- The face represented by (π, π_0) is nonempty if and only if $\max\{\pi x \mid x \in P\} = \pi_0$.
- If the face F is nonempty, we say it supports P.
- **3.1** Let P be a polyhedron with equality set $M^=$. If $F = \{x \in P | \pi^T x = \pi_0\}$ is nonempty, then F is a polyhedron. Let $M_F^= \supseteq M^=, M_F^\leq = M \setminus M_F^=$. Then $F = \{x \mid a_i^T x = b_i \forall i \in M_F^=, a_i^T x \leq b_i \forall i \in M_f^\leq\}$
  - We get the polyhedron F by taking some of the inequalities of P and making them equalities
  - The number of distinct faces of P is finite.

Facets

- A face F is said to be a facet of P if $\dim(F) = \dim(P) - 1$.
- Facets are all we need to describe polyhedra.
- **3.2** If F is a facet of P, then in any description of P, there exists some inequality representing F. (By setting the inequality to equality, we get F).
- **3.3 and 3.4** Every inequality that represents a face that is not a facet is unnecessary in the description of P.

POLLY's Faces

Consider the face

$$F = \{x \in POLLY \mid 2x_1 + 10x_2 - 5x_3 + 5x_4 - 3x_5 = 10\}$$

- Is it proper?
  - $F = POLLY$?
  - $F = \emptyset$?
- Points: $(0, 2, 2, 0, 0), (0, 1, 0, 0, 0), (0, 0, 0, 2, 0)$
  - Are they in F? (Don't forget, they must also be in P)
  - Are they affinely independent?
  - **Yes!** so $\dim(F) \geq 2$
  - Is $\dim(F) \leq 2$? (Yes!)
  - $\dim(POLLY) = 3$, so F is a facet of $POLLY$

Facet Representation

- Remember **3.2**. If F is a facet of $POLLY$, then there is some inequality $a_k^T x \leq b_k, k \in M^\leq$ representing F.
- Which inequality in the inequality set of $POLLY$ represents F?

- $x \in POLLY, 2x_1 + 10x_2 - 5x_3 + 5x + 4 - 3x_5 = 10 \Rightarrow x_1 = 0$
  - So F is represented by the inequality $-x_1 \leq 0$

Polyhedra—A Fundamental Representation Theorem

Putting together what we have seen so far, we can say the following: (**3.5**)

- Every full-dimensional polyhedron P has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of P.
- If $dim(P) = n - k$ with $k > 0$, then P is described by a maximal set of linearly independent rows of $(A^=, b^=)$, as well as one inequality representing each facet of P.

Polyhedra—A Useful Facet Proving Theorem

- Put another way, if a facet F of P is represented by (π, π_0), then the set of all representations of F is obtained by taking scalar multiples of (π, π_0) plus linear combinations of the equality set of P.
- We can use this to actually prove an inequality is a facet! (**3.6**)

- Let $M^= \equiv (A^=, b^=)$ be the equality set of $P \subseteq \mathbb{R}^n$, and let $F = \{x \in P \mid \pi^T x = \pi_0\}$ be a proper face of P. The following statements are equivalent
  - F is a facet of P
  - If $\lambda x = \lambda_0 \ \forall x \in F$, then

$$(\lambda, \lambda_0) = (\alpha\pi + uA^=, \alpha\pi_0 + u^t b^=),$$

for some $\alpha \in \mathbb{R}, u \in \mathbb{R}^{|M^=|}$

More Insight—Proving Facets

- This is just an indirect but very useful way to verify affine independence of points.
  - Here we assume that P is full dimensional $\dim(P) = n$ (though you can still use Theorem **3.6** even if not).
- Given valid inequality $\pi^T x \leq \pi_0$...
  1. Choose $t \geq n$ points $x^1, x^2 \ldots x^t$ all satisfying $\pi^T x = \pi_0$. Suppose that all these points also lie in a generic hyperplane $\lambda^T x = \lambda_0$.
  2. Solve the linear equation system:

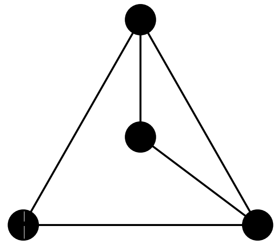
$$\sum_{j=1}^{n} \lambda_j x_j^k = \lambda_0 \ \forall k = 1, 2, \ldots t$$

  3. If the only solution is $(\lambda, \lambda_0) = \alpha(\pi, \pi_0)$ for $\alpha \neq 0$, then $\pi^T x \leq \pi_0$ is facet defining.

Example: The Node Packing Polytope

- Given a graph $G = (V, E)$, with $|V| = n$

$$\text{PACK}(G) = \{x \in \mathbb{B}^n \mid x_i + x_j \leq 1 \ \forall (i, j) \in E\}$$



$$\text{PACK} = \text{conv}\left(\begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right)$$

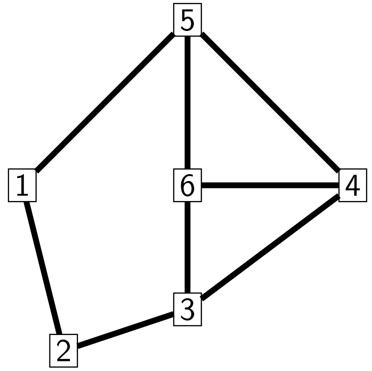
Example

Our Task

Prove that

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$$

is a facet-defining inequality for PACK($\hat{G}$):



Steps of the Proof

Your writing here

1. What is dim(PACK($\hat{G}$))?
2. So then rank$(A^=, b^=) = 0$
3. Let $F = \{x \in \text{PACK}(\hat{G}) \mid x_1 + x_2 + x_3 + x_4 + x_5 = 2\}$. Suppose $\lambda^T x = \lambda_0 \ \forall x \in F$. We will show that (λ, λ_0) is a scalar multiple of $[(1, 1, 1, 1, 1, 0)^T, 2]$.

An easier way?

- What we've just done is really just another way to show that the points we chose were affinely independent.

```
octave:1> A = [ 1, 0, 1, 0, 0, 0; 0, 1, 0, 1, 0, 0;
0, 0, 1, 0, 1, 0; 1, 0, 0, 1, 0, 0;
0, 1, 0, 0, 1, 0; 1, 0, 1, 0, 0, 1 ]

A =

  1  0  1  0  0  0
  0  1  0  1  0  0
  0  0  1  0  1  0
  1  0  0  1  0  0
  0  1  0  0  1  0
  1  0  1  0  0  1

octave:2> rank(A)
ans = 6
```


A More Abstract Example

- Let $C \subseteq V$ be a maximal clique in G. We will show (two ways) that

$$\sum_{i \in C} x_i \leq 1$$

- is a facet-defining inequality (a facet) of PACK(G).
- First question: What is dim(PACK)?
  - How do we establish the dimension of a polyhedron?

Way #1—Direct

- To show that $\sum_{i \in C} x_i \leq 1$ is a facet (that its dimension is $n-1$), we can given n affinely independent points in PACK that satisfy $\sum_{i \in C} x_i = 1$
  - Since the hyperplane $\sum_{i \in C} x_i = 1$ does not contain the origin, this is equivalent to giving n linearly independent points.
- WLOG, let the clique be $C = \{1, 2, \ldots, k\}$
- Key: $\forall p \in V \setminus C \; \exists i_p \in C$ such that $(i_p, p) \notin E$. **Why?**
- Points: $(e_1, e_2, \ldots, e_k, e_{k+1} + e_{i_{k+1}}, \ldots, e_p + e_{i_p}, \ldots, e_n + e_{i_n})$

Way #2—Indirect

- Let $F = \{x \in \text{PACK} | \sum_{i \in C} x_i = 1\}$
- Suppose $F \subseteq G \equiv \{x \in \text{PACK} | \lambda^T x = \lambda_0\}$ $(\lambda \neq 0)$
- If we can show that G is just a (non-zero) scalar multiple of F, then we have established that F is a facet.
- Again, WLOG, let $C = \{1, 2, \ldots, k\}$
- For $i \leq k$ consider the point e_i.
  - Satisfies equality F. $F \subseteq G \Rightarrow \lambda_i = \lambda_0 \; \forall i \in C$

Indirect Facet Proof, cont.

- For $p \in V \setminus C$, consider the point $e_p + e_{i_p}$: (1's in the coordinates p and i_p)
- By the same argument as the previous proof, this point packs, and we can always find such a point $\forall p \in V \setminus C$
- This point satisfies equality F. $F \subset G \Rightarrow \lambda_{i_p} + \lambda_p = \lambda_0$
- $\lambda_{i_p} = \lambda_0$, so $\lambda_p = \lambda_0 \; \forall p \in V \setminus C$.
- So out inequality defining G looks like $\lambda_0 \sum_{i \in C} x_i = \lambda_0$.
- This is a scalar multiple of the inequality defining F, so F is a facet defining inequality.
  - $\lambda_0 \neq 0$ since $\lambda \neq 0$, $\lambda_i = \lambda_0 \; \forall i \in C$, $\lambda_p = 0 \; \forall p \in V \setminus C$