IE418: Integer Programming

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Key Things We Learned Last Time

- A face F is said to be a facet of P if $dim(F) = dim(P) - 1$.
- All facets are necessary (and sufficient) to describe a polyhedron.
- Facets do not have a unique description. But...
  - Every full-dimensional polyhedron P has a unique (up to scalar multiplication) representation that consists of one inequality representing each facet of P.
  - If $dim(P) = n - k$ with $k > 0$, then P is described by a maximal set of linearly independent rows of $(A^=, b^=)$, as well as one inequality representing each facet of P.

Proving Facets—Way #2—Indirect

- This is just an indirect but very useful way to verify affine independence of points.
  - Here we assume that P is full dimensional $\dim(P) = n$ (though you can still use Theorem **3.6** even if not).
- Given valid inequality $\pi^T x \leq \pi_0$.
  1. Choose $t \geq n$ points $x^1, x^2 \ldots x^t$ all satisfying $\pi^T x = \pi_0$. Suppose that all these points also lie in a generic hyperplane $\lambda^T x = \lambda_0$.
  2. Solve the linear equation system:
  $$\sum_{j=1}^{n} \lambda_j x_j^k = \lambda_0 \ \forall k = 1, 2, \ldots t$$
  3. If the only solution is $(\lambda, \lambda_0) = \alpha(\pi, \pi_0)$ for $\alpha \neq 0$, then $\pi^T x \leq \pi_0$ is facet defining.

A More Abstract Example

$$\text{PACK}(G) = \{x \in \mathbb{B}^n \mid x_i + x_j \leq 1 \ \forall (i,j) \in E\}$$

- Let $C \subseteq V$ be a maximal clique in G. We will show (two ways) that

$$\sum_{i \in C} x_i \leq 1$$

- is a facet-defining inequality (a facet) of PACK(G).
- First question: What is dim(PACK)?
  - $|V|$!

Way #1—Direct

- To show that $\sum_{i \in C} x_i \leq 1$ is a facet (that its dimension is $|V| - 1$), we can given $|V|$ affinely independent points in PACK that satisfy $\sum_{i \in C} x_i = 1$
  - Since the hyperplane $\sum_{i \in C} x_i = 1$ does not contain the origin, this is equivalent to giving $|V|$ linearly independent points.
- WLOG, let the clique be $C = \{1, 2, \ldots, k\}$
- Key: $\forall p \in V \setminus C \ \exists i_p \in C$ such that $(i_p, p) \notin E$. **Why?**
- Points: $(e_1, e_2, \ldots, e_k, e_{k+1} + e_{i_{k+1}}, \ldots, e_p + e_{i_p}, \ldots, e_n + e_{i_n})$

Way #2—Indirect

- Let $F = \{x \in \text{PACK}(G) | \sum_{i \in C} x_i = 1\}$
- Suppose $F \subseteq H \stackrel{\text{def}}{=} \{x \in \text{PACK} | \lambda^T x = \lambda_0\}$ $(\lambda \neq 0)$
- If we can show that H is just a (non-zero) scalar multiple of F, then we have established that F is a facet.
- Again, WLOG, let $C = \{1, 2, \ldots, k\}$
- For $i \leq k$ consider the point e_i.
  - Satisfies equality F. $F \subseteq H \Rightarrow \lambda_i = \lambda_0 \ \forall i \in C$

Indirect Facet Proof, cont.

- For $p \in V \setminus C$, consider the point $e_p + e_{i_p}$: (1's in the coordinates p and i_p)
- By the same argument as the previous proof, this point packs, and we can always find such a point $\forall p \in V \setminus C$
- This point satisfies equality H. $F \subset G \Rightarrow \lambda_{i_p} + \lambda_p = \lambda_0$
- $\lambda_{i_p} = \lambda_0$, so $\lambda_p = 0 \ \forall p \in V \setminus C$.
- So our inequality defining H looks like $\lambda_0 \sum_{i \in C} x_i = \lambda_0$.
- This is a scalar multiple of the inequality defining F, so F is a facet defining inequality.
  - $\lambda_0 \neq 0$ since $\lambda \neq 0$, $\lambda_i = \lambda_0 \ \forall i \in C$, $\lambda_p = 0 \ \forall p \in V \setminus C$.

Extreme Points

- x is an extreme point of P if there do not exist $x^1, x^2 \in P$ such that $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$.
- x is an extreme point of P if and only if x is a zero-dimensional face of P.
- If (A, b) is a description of $P \neq \emptyset$ and $rank(A) = n - k$, then P has a face of dimension k and no proper face of lower dimension.
- These three results together imply that P has an extreme point if and only if $rank(A) = n$.
- This is the case for any polytope or any polyhedron lying in the nonnegative orthant.

Extreme Rays

- The recession cone P^0 associated with P is $\{r \in \mathbb{R}^n | Ar \leq 0\}$. Members of the recession cone are called rays of P.
- r is an extreme ray of P if there do not exist rays r^1 and r^2 of P such that $r = \frac{1}{2}r^1 + \frac{1}{2}r^2$.
- If $P \neq \emptyset$, then r is an extreme ray of P if and only if $\{\lambda r \mid \lambda \in \mathbb{R}_+\}$ is a one-dimensional face of P^0
- The last two results together imply that a polyhedron has a finite number of extreme points and extreme rays.
    - (Since there are a finite number of faces)

Good Ol' $POLLY$

- $POLLY \subseteq \mathbb{R}^5$:

$$
\begin{aligned}
x_1 - 2x_2 + x_3 - x_4 + 2x_5 &\leq 3 \\
x_1 - x_5 &\leq 0 \\
-x_1 + x_5 &\leq 0 \\
2x_2 - x_3 + x_4 &\leq 2 \\
-4x_2 + 2x_3 - 2x_4 &\leq -4 \\
3x_1 - x_2 &\leq 2 \\
-x_1 &\leq 0 \\
-x_2 &\leq 0 \\
-x_3 &\leq 0 \\
-x_4 &\leq 0 \\
-x_5 &\leq 0
\end{aligned}
$$

- $(1, 1, 0, 0, 1)^T$ is an extreme point of $POLLY$ **Prove it!**
- $(0, 1, 2, 0, 0)^T$ is an extreme ray of $POLLY$ **Prove it!**

Minkowski's Theorem

- If $P \neq \emptyset$ and $rank(A) = n$, then

$$P = \left\{ \sum_{k \in K} \lambda_k x^k + \sum_{j \in J} \mu_j r^j \mid \lambda_k \geq 0 \text{ for } k \in K, \mu_j \geq 0 \text{ for } j \in J, \sum_{k \in K} \lambda_i = 1 \right\}$$

- where $\{x^k\}_{k \in K}$ are the extreme points and $\{r^j\}_{j \in J}$ are the extreme rays.
- Corollaries
    - A nonempty polyhedron is bounded if and only if it has no extreme rays.
    - A polytope is the convex hull of its extreme points.
- A set of the form given above is called finitely generated.
- This result is often stated as "every polyhedron is finitely generated."

It's Still $POLLY$

- By Minkowski's Theorem, I can characterize $POLLY$ by her extreme points and extreme rays.

$$\text{ext}(POLLY) = \left\{ \begin{pmatrix} 5/3 \\ 3 \\ 4 \\ 0 \\ 5/3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2/3 \\ 0 \\ 0 \\ 2 \\ 2/3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\text{cone}(POLLY) = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

- $\lambda_2 = 1/3, \lambda_4 = 2/3, \mu_2 = 23$
- $\hat{x} = (0, 71/3, 23, 2/3, 0) \in POLLY$

Converting From One Description to Another

- In *theory*, we can always convert from the inequality description of a polyhedron to the extreme-point-extreme-ray description.
- This could be (and is) *very* useful when trying (for example) to determine valid inequalities for a class of integer programs. **Why?**
- In *practice*, how can we convert from one description to another?
  - "Double Description Algorithm"
  - "Fourier-Motzkin Elimination".
  - Programs: PORTA, Polymake, lrs, ddd, etc...
  - Show and Tell! (time permitting)
- Buyer Beware— "small" extreme point descriptions can lead to *huge* number of inequalities and vice versa

Polymake

- On shark in `/usr/local/polymake`
- Wiki entry `http://coral.ie.lehigh.edu/cgi-bin/wiki.pl?PolymakeInformation`
- Inequalities are all of form: $a_0 + a_1x_1 + \ldots + a_nx_n \geq 0$
- "Points" are given such that first column is '1', then it is a extreme point, otherwise, it is an extreme ray.

- *POLLY*

```
INEQUALITIES
3 -1 2 -1 1 -2
0 -1 0 0 0 1
0 1 0 0 0 -1
2 0 -2 1 -1 0
-4 0 4 -2 2 0
2 -3 1 0 0 0
0 1 0 0 0 0
0 0 1 0 0 0
0 0 0 1 0 0
0 0 0 0 1 0
0 0 0 0 0 1
```


Results from Linear Programming

Define the following:

- $P = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$, $z = \max\{cx \mid x \in P\}$
- $\mathcal{Q} = \{u \in \mathbb{R}^m_+ \mid uA \geq c\}$, $w = \min\{ub \mid u \in \mathcal{Q}\}$
- $\{x^k\}_{k \in K}$, $\{u^i\}_{i \in I}$ are the extreme points of P and $\mathcal{Q}$ respectively.
- $P^0 = \{x \in \mathbb{R}^n_+ \mid Ax \leq 0\}$
- $Q^0 = \{u \in \mathbb{R}^m_+ \mid u^T A \geq 0\}$
- $\{r^j\}_{j \in J}$, $\{v^t\}_{t \in T}$ are the extreme rays of P^0 and $\mathcal{Q}^0$ respectively.

Results from LP

- $P \neq \emptyset \Leftrightarrow vb \geq 0 \; \forall t \in T$
- The following are equivalent when $P \neq \emptyset$:
  1. z is unbounded from above,
  2. there exists an extreme ray r^j of P with $cr^j > 0$,
  3. $\mathcal{Q} = \emptyset$
- If $P \neq \emptyset$ and z is bounded, then

$$z = \max_{k \in K} cx^k = w = \min_{i \in I} u^i b$$

Projections and Polyhedra

- If $p \in \mathbb{R}^n$ and H is a subspace, the projection of p onto H is the vector $q \in H$ such that $p - q \in H^{\perp}$.
- Note that this is a decomposition of a vector p into the sum of a vector in H and a vector in $H^{\perp}$.
- The *projection of a set* is the union of the projections of all its members.
- We will often be interested in "projecting out" a set of variables, i.e., projecting P into a subspace $\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid y = 0\}$.
- The projection of a point (x, y) into this subspace is the point $(x, 0)$.

The Projection of a Polyhedron

- Let $P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid Ax + Gy \leq b\}$
- So the projection of P into the space of just the x variables is

$$\begin{aligned} \text{proj}_x(P) &= \{x \in \mathbb{R}^n \mid (x, 0) \in P\} \\ &= \{x \in \mathbb{R}^n \mid v^T(b - Ax) \geq 0 \ \forall t \in T\} \end{aligned}$$

  where $\{v^t\}_{t \in T}$ are the extreme rays of $Q = \{v \in \mathbb{R}^M_+ \mid vG \geq 0\}$.
- This immediately implies that the projection of a polyhedron is a polyhedron.

Weyl's Theorem

If

$$Q = \left\{ \sum_{k \in K} \lambda_k x^k + \sum_{j \in J} \mu_j r^j \mid \lambda_k \geq 0 \text{ for } k \in K, \mu_j \geq 0 \text{ for } j \in J, \sum_{k \in K} \lambda_i = 1 \right\}$$

where $\{x^k\}_{k \in K}$ and $\{r^j\}_{j \in J}$ are given sets of rational vectors, then Q is a rational polyhedron.

- This is the converse of Minkowski's Theorem.
- This says roughly "every finitely generated set is a polyhedron"
- The proof is easy using projection (Read it in the book)...