IE418: Integer Programming

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Today's Outline

- Some Problems
  - Separation Problem
  - Optimization Problem
- The Ellipsoid Method
  - An informal introduction
- The equivalance of separation and optimization

The Separation Problem

- Consider the following problem
  $OP : \max\{c^T x \mid x \in X \subseteq \Re^n\}$.
- The **Separation Problem** (SEP) associated with OP is the following:
  - Given $\hat{x} \in \Re^n$, is $\hat{x} \in \text{conv}(X)$?
  - If not, give a "certificate"— An inequality $\pi^T x \leq \pi_0$ satisfied by all points in X, but violated by $\hat{x}$: $(\pi^T \hat{x} > \pi_0)$.
- Our goal in the following will be to show (loosely) that solving the separation problem in polynomial time is equivalent to solving the optimization problem in polynomial time

The Ellipsoid Algorithm

- Due to Khachiyan in 1979
  - Building on the work of Shor, Yudin, and Nemirovskii
- The *first* algorithm that demonstrated that linear programming was solvable in polynomial time
- Now many interior point algorithms have polynomial complexity
- The Ellipsoid method is considered to be computationally impractical
- Its consequences in combinatorial optimization are enormous!
- In what follows, we will assume that
  - P is bounded.
  - P is full-dimensional

Ellipsoid Algorithm

1. Find ellipsoid $E_0 \supseteq P$
2. Find center x_0 of E_0
3. Test if $x_0 \in P$.
4. If $x_0 \in P$, you are done. Otherwise, find the violated inequality (π, π_0).
5. Push the (π, π_0) until it hits x_0, giving you a half-ellipsoid HE that contains P.
6. Find a new ellipsoid $E_1 \supseteq HE$ such that

$$\frac{\text{volume}(E_1)}{\text{volume}(E_0)} \leq e^{-1/(2n)} < 1$$

7. $E_0 \leftarrow E_1$. Go to 2.

Ellipsoid Method

- The shrinking is geometric (multiplicative)...
- In a polynomial number of steps, you either show
  - A point $\hat{x}$ in P
  - P is empty
  - The volume gets smaller than a lower bound (so P is empty within tolerance).
- Given a linear program, if we can solve the separation problem in polynomial time, then we can solve the optimization problem in polynomial time using the ellipsoid method.

Minimum Cut (MCP)

- Given $G = (V, E)$, $s, t, \in V$, let $\mathcal{P}$ be the collection of all $s - t$ paths in G. The problem of finding a minimum $s - t$ cut in $G = (V, E)$ is

$$\min_{e \in E} c_e x_e$$

subject to

$$\begin{aligned} \sum_{e \in P} x_e &\geq 1 \qquad \forall P \in \mathcal{P} \\ 0 \leq x_e &\leq 1 \qquad \forall e \in E \end{aligned}$$

- Solving this *linear* program gives you a minimum $s - t$ cut.
- This isn't obvious, since it is not clear that the extreme points are integral.

Separation Example, Cont.

- Can I solve the LP relaxation of the MCP problem in polynomial time?
- **There are an exponential number of inequalities!**
  - So even if our LP algorithm runs in time polynomial in the number of inequalities, this is not a polynomial algorithm!
- Thank goodness for the Ellipsoid Method
- Given $\hat{x} \in \Re^{|E|}$, can I check (in polynomial time) whether or not $\sum_{e \in P} \hat{x}_e \geq 1 \quad \forall P \in \mathcal{P}$?
- Find a minimum weight $s - t$ path, with weights $\hat{x}$!

TSP-LP

$$\min \sum_{e \in E} c_e x_e$$

subject to

$$\begin{array}{rcll} \sum_{e \in \delta(\{v\})} x_e & \leq & 2 & \forall v \in V \\ \sum_{e \in \delta(U)} x_e & \geq & 2 & \forall U \subset V \mid 3 \leq |U| \leq |V|/2 \end{array}$$

- $\forall e = (s,t) \in E$ solve

$$\min \sum_{e \in \delta(U)} \hat{x}_e \mid U \subset V s \in U, t \in V \setminus U$$

- (As far as I know) there is no known polynomial algorithm for solving this problem *other* than the ellipsoid method.

Wrap-up

- By ellipsoid algorithm, if you can separate in polynomial time, then you can solve the optimization problem in polynomial time
- Key now—Show that if you can solve an LP in polynomial time, then you must be able to solve its separation problem in polynomial time. For that we need polarity.

Polarity

- I am assuming you all read the section on Polarity. (I.4.5)
- Here are the key results we need
- If $P = \{x \in \Re^n | Ax \leq b\}$,
  $\Pi = \{(\pi, \pi_0) \in \Re^{n+1} | \pi^T x \leq \pi_0 \ \forall x \in P\}$ is the **polar** of P.
- Let $P \subseteq \Re^n$ be a polyhedron with extreme points $\{x^k\}_{k \in K}$ and extreme rays $\{r_j\}_{j \in J}$. Then $\Pi = \{(\pi, \pi_0)\}$ is the following polyhedral cone:

$$\begin{array}{rcll} \pi^T x^k - \pi_0 & \leq & 0 & \forall k \in K \\ \pi^T r^j & \leq & 0 & \forall j \in J \end{array}$$

Example

$$P = \{x \in \mathbb{R}^2 \mid x_1 + x_2 \leq 1, -x_1 \leq 0, -x_2 \leq 0\}$$

- Is $(1, 2, 4) \in \Pi$?
- Does $x_1 + 2x_2 \leq 4 \forall x \in P$?
- $\text{ext}(P) = \{(0,0,0)^T, (0,1,0)^T, (0,0,1)^T\}$

$$\Pi = \{(\pi, \pi_0) \in \mathbb{R}^3 \mid -\pi_0 \leq 0, \pi_1 - \pi_0 \leq 0, \pi_2 - \pi_0 \leq 0\}$$

More Cool Polarity Stuff

- Assume $\dim(P) = \text{rank}(A) = n$

Cool!

- The facets of P are the extreme rays of the polar of P!
- (π, π_0) is an extreme ray of Π iff (π, π_0) is a facet of P.

And Vice Versa!

- $\pi^T \hat{x} \leq \pi_0$ defines a facet of Π if and only if $\hat{x}$ is an extreme point of P
- $\pi^T \hat{r} \leq 0$ defines a facet of Π if and only if $\hat{r}$ is an extreme ray of P

The 1-Polar

- Let's assume we are dealing with polytopes.
- If $\hat{P} = \{x \in \Re^n \mid Ax \leq b\}$ is full dimensional it has an interior point.
- By translation, we can take this interior point to be 0.
- So if $a^T x \leq b$ is valid inequality for the (translated) $\hat{P}$, $b > 0$
- You can scale each inequality by the RHS and rewrite the polytope as $P = \{x \in \Re^n \mid Ax \leq 1\}$
- The **1-polar** of P is $\Pi^1 = \{\pi \in \Re^n \mid \pi^T x^k \leq 1 \ \forall k \in K \equiv \text{ext}(P)\}$

Main Theorem

- If P is full dimensional and 0 is an interior point of P, then
- $P = \{x \mid x^T \pi^t \leq 1 \ \forall t \in T \equiv \text{ext}(\Pi^1)\}$
- $\Pi^1 = \{\pi \mid \pi^T x^k \leq 1 \ \forall k \in K \equiv \text{ext}(P)\}$
- The consequence of this is the following:
- $\hat{x} \in P \Leftrightarrow \max\{\hat{x}^T \pi | \pi \in \Pi^1\} \leq 1$
- $\hat{\pi} \in \Pi^1 \Leftrightarrow \max\{\hat{\pi}^T x | x \in P\} \leq 1$

Separation is Equivalent to Optimization

- Separate over P in $\mathcal{P}$ $\overset{\text{Ellipsoid}}{\Rightarrow}$ Solve LP over P in $\mathcal{P}$
- Solve LP over P in $\mathcal{P}$ $\overset{\text{Polarity}}{\Rightarrow}$ Separate over Π in $\mathcal{P}$
- Separate over Π in $\mathcal{P}$ $\overset{\text{Ellipsoid}}{\Rightarrow}$ Solve LP over Π in $\mathcal{P}$
- Solve LP over Π in $\mathcal{P}$ $\overset{\text{Polarity}}{\Rightarrow}$ Separate over P in $\mathcal{P}$ Q.E.D