IE418: Integer Programming

Jeff Linderoth

Department of Industrial and Systems Engineering
Lehigh University

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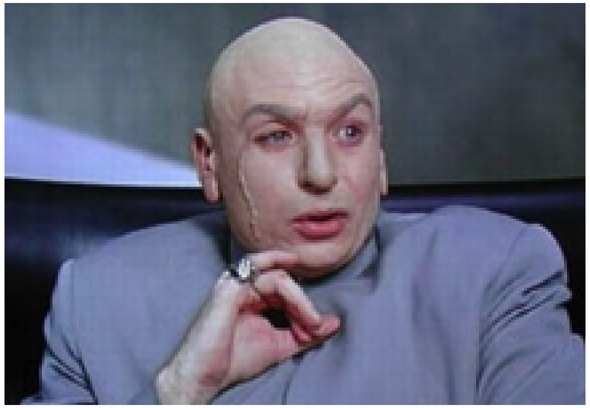
Evil!

- Hand in Homeworks!



Valid Inequalities for the Knapsack Problem

- We are interested in valid inequalities for the knapsack set KNAP

$$\text{KNAP} = \{x \in \mathbb{B}^n | \sum_{j \in N} a_j x_j \leq b\}$$

- $N = \{1, 2, \ldots n\}$
- We assume WLOG that $a_j > 0 \; \forall j \in N$ Why?
  - If $a_j < 0$, let $\hat{x_j} = 1 - x_j$
- We will also assume that $a_j < b \; \forall j \in N$
- We are interested in finding facets of conv(KNAP)

Using Valid Inequalities for a Relaxation

- I want to solve MIPs, why do I care about strong inequalities for the knapsack problem?
- If $P = \{x \in \mathbb{B}^n \mid Ax \leq b\}$, then for any row i, $P_i = \{x \in \mathbb{B}^n \mid a_i^T x \leq b_i\}$ is a relaxation of P.
  - $P \subseteq P_i \; \forall i = 1, 2, \ldots m$
  - $P \subseteq \bigcap_{i=1}^m P_i$
- Any inequality valid for a relaxation of an IP is valid for the IP itself.
- Generating valid inequalities for a relaxation is often easier.
- If the intersection of the relaxations is a good approximation to the true problem, then the inequalities will be quite useful.
- Crowder, Johnson, and Padberg is the seminal paper that shows this to be true.

Simple facets

- What is dim(conv(KNAP))?
  - $0, e_j, \forall j \in N$ are $n+1$ affinely independent points in conv(KNAP) $\Rightarrow$ dim(conv(KNAP)) $= n$.
- $x_k \geq 0$ is a facet of conv(KNAP)
  - **Proof.** $0, e_j, \ \forall j \in N \setminus k$ are n affinely independent points that satisfy $x_k = 0$
- $x_k \leq 1$ is a facet of conv(KNAP) if $a_j + a_k \leq b \ \forall j \in N \setminus k$
  - **Proof.** $e_k, e_j + e_k, \ \forall j \in N \setminus k$ are n affinely independent points that satisfy $x_k = 1$.

Covers

- A set $C \subseteq N$ is a *cover* if $\sum_{j \in C} a_j > b$
- A cover C is a *minimal cover* if $C \setminus j$ is not a cover $\forall j \in C$
- If $C \subseteq N$ is a cover, then the *cover inequality*

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a valid inequality for S

Example

$$\text{MYKNAP} = \{x \in \mathbb{B}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$$

- Some minimal covers are the following:

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 2 \\ x_1 + x_2 + x_6 &\leq 2 \\ x_1 + x_5 + x_6 &\leq 2 \\ x_3 + x_4 + x_5 + x_6 &\leq 3 \end{aligned}$$

Best We Can Do?

- Are these inequalities the strongest ones we can come up with?
- What does *strongest* mean?
  - We all know that facets are the "strongest", but can we say anything else
- If $\pi^T x \leq \pi_0$ and $\mu^T x \leq \mu_0$ are two valid inequalities for $P \subseteq \Re^n_+$, we say that $\pi^T x \leq \pi_0$ **dominates** $\mu^T x \leq \mu_0$ if $\exists u \geq 0$ such that
  - $\pi \geq u\mu$
  - $\pi_0 \leq u\mu_0$
  - $(\pi, \pi_0) \neq u(\mu, \mu_0)$
- If $\pi^T x \leq \pi_0$ dominates $\mu^T x \leq \mu_0$, then $\{x \in \Re^n_+ \mid \pi^T x \leq \pi_0\} \subseteq \{x \in \Re^n_+ \mid \mu^T x \leq \mu_0\}$

Back to the Knapsack

- If $C \subseteq N$ is a minimal cover, the *extended cover* $E(C)$ is defined as
  - $E(C) = C \cup \{j \in N \mid a_j \geq a_i \ \forall i \in C\}$
- If $E(C)$ is an extended cover for S, then the *extended cover inequality*

$$\sum_{j \in E(C)} x_j \leq |C| - 1,$$

  is a valid inequality for S
- **Proof.** $x^R \in \text{KNAP}, \sum_{j \in E(C)} x_j^R \geq |C| \Rightarrow |R \cap E(C)| \geq |C|$. $\sum_{j \in R} a_j \geq \sum_{j \in R \cap E(C)} a_j \geq \sum_{j \in C} a_j > b$, so $x^R \notin \text{KNAP}$
- Note this inequality dominates the cover inequality if $E(C) \setminus C \neq \emptyset$
- **(Example, cont.)** The cover inequality $x_3 + x_4 + x_5 + x_6 \leq 3$ is dominated by the extended cover inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$

As Good As It Gets?

- **Question:** Are these inequalities as strong as possible?
- **Answer:** Sometimes.
- Order the variables so that $a_1 \geq a_2 \ldots \geq a_n$
- Denote the cover as $C = \{j_1, j_2, \ldots j_r\}$ $(j_1 < j_2 < \ldots < j_r)$ so that $a_{j_1} \geq a_{j_2} \geq \ldots \geq a_{j_r}$
- Given a minimal cover C, define the following sets:
  - $R_k = C \setminus k \ \forall k \in C$
  - $S_k = C \setminus \{j_1, j_2\} \cup \{k\} \ \forall k \in E(C) \setminus C$
  - $T_k = C \setminus j_1 \cup \{k\} \ \forall k \in N \setminus E(C)$

Facets

- If $C = N$, $\sum_{j \in C} x_j \leq |C| - 1$ is a facet of conv(KNAP)
  - **Proof.** R_k.
- Let $p = \min\{j \mid j \in N \setminus E(C)\}$. If $C = E(C)$, and $\sum_{j \in C \setminus j_1} a_j + a_p \leq b$, then $\sum_{j \in C} x_j \leq |C| - 1$ is a facet of conv(KNAP).
  - **Proof.** R_k and T_k.
- If $E(C) = N$ and $\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b$, then $\sum_{j \in E(C)} x_j \leq |C| - 1$ is a facet of conv(KNAP).
  - **Proof.** R_k and S_k.

In General...

- Order the variables so that $a_1 \geq a_2 \ldots \geq a_n$
- Let C be a cover with $C = \{j_1, j_2, \ldots j_r\}$ $(j_1 < j_2 < \ldots < j_r)$ so that $a_{j_1} \geq a_{j_2} \geq \ldots \geq a_{j_r}$. Let $p = \min\{j \mid j \in N \setminus E(C)\}$.
- If any of the following conditions hold, then

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

  gives a facet of conv(KNAP)
  - $C = N$
  - $E(C) = N$ and (*) $\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b$
  - $C = E(C)$ and (**) $\sum_{j \in C \setminus j_1} a_j + a_p \leq b$
  - $C \subset E(C) \subset N$ and (*) and (**).

Examples

- $C = \{1, 2, 6\}$. $E(C) = C$.
  - If $a_2 + a_6 + a_3 \leq b$, then $x_1 + x_2 + x_6 \leq 2$ is a facet of conv(MYKNAP)
  - $16 \leq 19$. It is a facet!
- $C = \{3, 4, 5, 6\}$. $E(C) = \{1, 2, 3, 4, 5, 6\}$. $C \subset E(C) \subset N$. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$ is a facet of conv(MYKNAP) if...
  - $a_4 + a_5 + a_6 + a_7 \leq b$? **(Yes!)**
  - $a_5 + a_6 + a_1 \leq b$ **(No!)**,
- So $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$ is **not** facet-defining for conv(MYKNAP)

conv(MYKNAP)

- From out friend polymake...

$$
\begin{array}{rcll}
x_j & \geq & 0 & \forall j = 1, 2, \ldots, 7 \\
x_j & \leq & 1 & \forall j = 1, 2, \ldots, 7 \\
x_1 + x_5 + x_6 & \leq & 2 & \\
x_1 + x_4 + x_6 & \leq & 2 & \\
x_1 + x_4 + x_5 & \leq & 2 & \\
x_1 + x_3 + x_6 & \leq & 2 & \\
x_1 + x_3 + x_5 & \leq & 2 & \\
x_1 + x_3 + x_4 & \leq & 2 & \\
x_1 + x_2 + x_6 & \leq & 2 & \\
x_1 + x_2 + x_5 & \leq & 2 & \\
x_1 + x_2 + x_4 & \leq & 2 & \\
x_1 + x_2 + x_3 & \leq & 2 & \\
2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 & \leq & 3 &
\end{array}
$$

Covers and Lifting

- Let $P_{1,2,7} = \text{conv}(\text{MYKNAP} \cap \{x \in \Re^7 \mid x_1 = x_2 = x_7 = 0\})$
- Consider the cover inequality arising from $C = \{3, 4, 5, 6\}$.
- $\sum_{j \in C} x_j \leq 3$ is facet defining for $P_{1,2,7}$
- If x_1 is not fixed at 0, can we strengthen the inequality?
- For what values of α_1 is the inequality

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

valid for

$$P_{2,7} = \text{conv}(\{x \in \text{MYKNAP} \mid x_2 = x_7 = 0\})?$$

- If $x_1 = 0$ then the inequality is valid for all values of α_1

The Other Case

- If $x_1 = 1$, the inequality is valid if and only if

$$\alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

is valid for all $x \in \mathbb{B}^4$ satisfying

$$6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11$$

- Equivalently, if and only if

$$\alpha_1 + \max_{x \in \mathbb{B}^4} \{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\} \leq 3$$

- Equivalently if and only if $\alpha_1 \leq 3 - \gamma$, where

$$\gamma = \max_{x \in \mathbb{B}^4} \{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}.$$

Solving the Knapsack Problem

- In this case, we can "solve" the knapsack problem to see that $\gamma = 1$. Therefore $\alpha_1 \leq 2$.
- The inequality

$$2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

is a valid inequality for P_{27}

  - Is it facet-defining?

Lifting

- What we've just done is called *lifting*. Where a valid (and facet defining) inequality for $S \cap \{x \in \mathbb{B}^n \mid x_k = 0\}$ is turned into a facet defining inequality for S.
- **Theorem.** Let $S \subseteq \mathbb{B}^n$, for $\delta \in \{0,1\}, S^\delta = S \cap \{x \in \mathbb{B}^n \mid x_1 = \delta\}$. Suppose

$$\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

is valid for S^0.

Lifting Thm. (2)

- If $S^1 = \emptyset$, then $x_1 \leq 0$ is valid for S
- If $S^1 \neq \emptyset$, then

$$\alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

is valid for S for any $\alpha_1 \leq \pi_0 - \gamma$, where

$$\gamma - \max\{\sum_{j=2}^{n} \pi_j x_j \mid x \in S^1\}.$$

Lifting Thm. (3)

- If $\alpha_1 = \pi_0 - \gamma$ and $\sum_{j=2}^{n} \pi_j x_j \leq \pi_0$ defines a face of dimension k of conv(S^0), then

$$\alpha_1 x_1 + \sum_{j=2}^{n} \pi_j x_j \leq \pi_0$$

defines a face of dimension *at least* $k + 1$ of conv(S).

For You To Do...

- Read N&W Sections II.2.1, II.2.2
- Read [2]
- Read [1]

A. Atamtürk, *Cover and pack inequalities for (mixed) integer programming*, Annals of Operations Research. forthcoming.

H. Crowder, E. L. Johnson, and M. W. Padberg, *Solving large scale zero-one linear programming problems*, Operations Research, 31 (1983), pp. 803–834.