IE418: Integer Programming

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Last Time...

- conv(KNAP) = conv($\{x \in \mathbb{B}^n | \sum_{j \in N} a_j x_j \leq b\}$)
- $C \subseteq N \mid \sum_{j \in C} a_j > b$, $\sum_{C \setminus k} a_j \leq b \forall k \in C$
- For "extended" cover $E(C)$ if any of the following conditions hold, then

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

gives a facet of conv(KNAP)
  - $C = N$
  - $E(C) = N$ and (*) $\sum_{j \in C \setminus \{j_1, j_2\}} a_j + a_1 \leq b$
  - $C = E(C)$ and (**) $\sum_{j \in C \setminus j_1} a_j + a_p \leq b$
  - $C \subset E(C) \subset N$ and (*) and (**).

An "Uplifting" Experience

- $S \subseteq \mathbb{B}^n$
- Lifting is a process in which a valid (and facet defining) inequality for $S \cap \{x \in \mathbb{B}^n \mid x_k = 0\}$ is turned into a facet defining inequality for S.
- **Theorem.** Let $S \subseteq \mathbb{B}^n$, for $\delta \in \{0, 1\}, S^\delta = S \cap \{x \in \mathbb{B}^n \mid x_1 = \delta\}$. Suppose

$$\sum_{j=2}^n \pi_j x_j \leq \pi_0$$

is valid for S^0.

Lifting Thm. (2)

- If $S^1 = \emptyset$, then $x_1 \leq 0$ is valid for S
- If $S^1 \neq \emptyset$, then $\alpha_1 x_1 + \sum_{j=2}^n \pi_j x_j \leq \pi_0$ is valid for S for any $\alpha_1 \leq \pi_0 - \gamma$, where

$$\gamma - \max\{\sum_{j=2}^n \pi_j x_j \mid x \in S^1\}.$$

- If $\alpha_1 = \pi_0 - \gamma$ and $\sum_{j=2}^n \pi_j x_j \leq \pi_0$ defines a face of dimension k of conv(S^0), then

$$\alpha_1 x_1 + \sum_{j=2}^n \pi_j x_j \leq \pi_0$$

defines a face of dimension *at least* $k + 1$ of conv(S).

Uplifting Example

- Let $P_{1,2,7} = \text{conv}(\text{MYKNAP} \cap \{x \in \mathbb{R}^7 \mid x_1 = x_2 = x_7 = 0\})$
- Consider the cover inequality arising from $C = \{3, 4, 5, 6\}$.
- $\sum_{j \in C} x_j \leq 3$ is facet defining for $P_{1,2,7}$
- If x_1 is not fixed at 0, can we strengthen the inequality?
- For what values of α_1 is the inequality

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

valid for

$$P_{2,7} = \text{conv}(\{x \in \text{MYKNAP} \mid x_2 = x_7 = 0\})?$$

- If $x_1 = 0$ then the inequality is valid for all values of α_1

Uplifting Example (2)

- If $x_1 = 1$, the inequality is valid if and only if

$$\alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

is valid for all $x \in \mathbb{B}^4$ satisfying

$$6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11$$

- Equivalently, if and only if

$$\alpha_1 + \max_{x \in \mathbb{B}^4}\{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\} \leq 3$$

- Equivalently if and only if $\alpha_1 \leq 3 - \gamma$, where

$$\gamma = \max_{x \in \mathbb{B}^4}\{x_3 + x_4 + x_5 + x_6 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8\}.$$

You Can Also "DownLift"

- $s \subseteq \mathbb{B}^n, S^1 = S \cap \{x \in \mathbb{B}^n \mid x_1 = 1\}$
- Let $\sum_{j=2}^n \pi_j x_j \leq \pi_0$ be valid for S^1.
- If $S^0 = \emptyset, x_1 \geq 1$ is valid for S, otherwise

$$\xi_1 x_1 + \sum_{j=2}^n \pi_j x_j \leq \pi_0 + \xi_1$$

is valid for S, for $\xi_i \geq \gamma - \pi_0$

  - $\gamma = \max\{\sum_{j=2}^n \pi_j x_j \mid x \in S^0\}$.
- Similar facet/dimension results to uplifting if the lifting is maximum.

DownLifting Example

- Let $P_6^1 = \text{conv}(\text{MYKNAP} \cap \{x \in \Re^7 \mid x_6 = 1\})$
- **Fact:** $x_1 + x_5 \leq 1$ is facet-defining for P_6^1.
  - $C = E(C)$ and $\sum_{j \in C \setminus j_1} a_j + a_p \leq b$
  - **Note:** $x_1 + x_5 \leq 1$ is not valid for MYKNAP
- For what values of α is the inequality $x_1 + x_5 + \alpha(x_6 - 1) \leq 1$ valid for MYKNAP?
- If $x_6 = 1$, then valid if $\alpha \in [-\infty, \infty]$
- If $x_6 = 0$, then valid if $\alpha \geq x_1 + x_5 - 1 \; \forall x \in \text{MYKNAP}$
- If and only if $\alpha \geq \max_{x \in \mathbb{B}^7}\{x_1 + x_5 - 1 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + x_7 \leq 19\}$
- $\alpha \geq 1$.
- $x_1 + x_5 + x_6 \leq 2$ is valid *and facet-defining* inequality for MYKNAP.

General Lifting and SuperAdditivity

- $K = \text{conv}(\{x \in \mathbb{Z}_+^{|N|}, y \in \Re_+^{|M|} \mid a^T x + g^T y \leq b, x \leq u\})$
- Partition N into $[L, U, R]$
  - $L = \{i \in N \mid x_i = 0\}$
  - $U = \{i \in N \mid x_i = u_i\}$
  - $R = N \setminus L \setminus U$
- We will use the notation: x_R to mean the vector of variables that are in the set R.
  - $a_R^T x_R = \sum_{j \in R} a_j x_j$

$$K(L, U) = \text{conv}(\{x \in \mathbb{Z}_+^{|N|}, y \in \Re_+^{|M|} \mid$$
$$a_R^T x + g^T y \leq d, x_R \leq u_R, x_i = 0 \ \forall i \in L, x_i = u_i \ \forall i \in U.\})$$

- So $d = b - a_U^T x_U$

Lifting

- Let $\pi^T x_R - \sigma^T y \leq \pi_0$ be a valid inequality for $K(L, U)$.
- Consider the *lifting function* $\Phi : \Re \to \Re \cup \{\infty\}$
  - (∞) if lifting problem is infeasible

$$\Phi(\alpha) = \pi_0 - \max\{\pi_R^T x_R + \sigma^T y \mid$$
$$a_R^T x_R + g^T y \leq d - \alpha, x_R \leq u_R, x_R \in \mathbb{Z}_+^{|R|}, y \in \Re_+^{|M|}\}$$

- In words, $\Phi(\alpha)$ is the maximum value of the LHS of the valid inequality if the RHS in K is reduced by α.

Φ, Schmi

- Why do we care about Φ?

$$\pi_R^T x_R + \pi_L^T x_L + \pi_U^T (u_U - x_U) + \sigma^T y \leq \pi_0$$

is a valid inequality for K if and only if

$$\pi_L^T x_L + \pi_U^T (u_U - x_U) \leq \Phi(a_L^T x_L + a_U^T (x_U - u_U)) \ \forall (x, y) \in K.$$

Proof.?

Example—Sequential Lifting

- Lifting one variable (at a time) in 0-1 IP (like we have done so far)...
- $\alpha x_k + \pi_R^T x_R \leq \pi_0$ is valid for $P \Leftrightarrow \alpha x_k \leq \Phi(a_k x_k) \ \forall x \in P$
  - $x_k = 0, \quad 0 \leq \Phi(0)$ is always true.
  - $x_k = 1, \quad \Rightarrow \alpha \leq \Phi(a_l)$
- If I "know" $\Phi(q) (\forall q \in \Re)$, I can just "lookup" the value of the lifting coefficient for variable x_k
- Note that if I have restricted more than one variable, then this "lookup" logic is not necessarily true
  - For lifting two (0-1) variables, I would have to look at four possible values.
  - In general, the lifting function changes with each new variable "lifted".

Superadditivity

- A function $\phi : \Re \to \Re$ is *superadditive* if

$$\phi(q_1) + \phi(q_2) \leq \phi(q_1 + q_2)$$

- Superadditive functions play a significant role in the theory of integer programming. (See N&W page 229). (We'll probably revisit them later).
- Superadditive Fact:

$$\sum_{j \in N} \phi(a_j) x_j \leq \sum_{j \in N} \phi(a_j x_j) \leq \phi\left(\sum_{j \in N} a_j x_j\right).$$

"Multiple Lookup"—Superadditivity

- Suppose that ϕ is a superadditive lower bound on Φ that satisfies $\pi_i = \phi(a_i)\ \forall i \in L$ and $\pi_i = \phi(-a_i)\ \forall i \in U$

$$\begin{aligned} \sum_{i \in L} \phi(a_i) x_i + \sum_{i \in U} \phi(-a_i)(u_i - x_i) &\leq \phi(a_L^T x_L + a_U^T (x_U - u_U)) \\ &\leq \Phi(a_L^T x_L + a_U^T (x_U - u_U)) \end{aligned}$$

- So

$$\pi_R^T x_R + \pi_L^T x_L + \pi_U^T (u_U - x_U) + \sigma^T y \leq \pi_0$$

is a valid inequality for K

The Main Result

- If ϕ is a superadditive lower bound on Φ, any inequality of the form $\pi_R^T x_R - \sigma^T y \leq \pi_0$, which is valid for $K(L, U)$, can be extended to the inequality

$$\pi_R^T x_R + \sum_{j \in L} \phi(a_j) x_j + \sum_{j \in U} \phi(-a_j)(u_j - x_j) + \sigma^T y \leq \pi_0$$

which is valid for K.
- If $\pi_i = \phi(a_i)\ \forall i \in L$ and $\pi_i = \phi(-a_i)\ \forall i \in U$ and $\pi^T x_R - \sigma^T y = \pi_0$ defines a k-dimensional face of $K(L, U)$, then the lifted inequality defines a face of dimension at least $k + |L| + |U|$.

This Is Soooooooooo Cool

- What does this imply?
- If the lifting function itself is superadditive, I can lift *all* of the variables in one pass (if I know the lifting function, of course).
- Even if I don't know the lifting function, if I can get a superadditive function that is a lower bound, then I can lift all the variables at once.

Example

- This treatment follows that of Atamtürk's paper I handed out.
- Cover C with $\lambda = a(C) - b > 0$
  - Write our knapsack cover inequality as

$$\sum_{j \in C} \lambda x_j \leq \lambda(|C| - 1)$$

- Lifting function $\Theta : \Re \rightarrow \Re \cup \{\infty\}$

$$\Theta(\alpha) = \lambda(|C| - 1) - \max\{\sum_{j \in C} \lambda x_j \mid \sum_{j \in C} a_j x_j \leq b - \alpha\}.$$

Example—Lifted Knapsack Covers

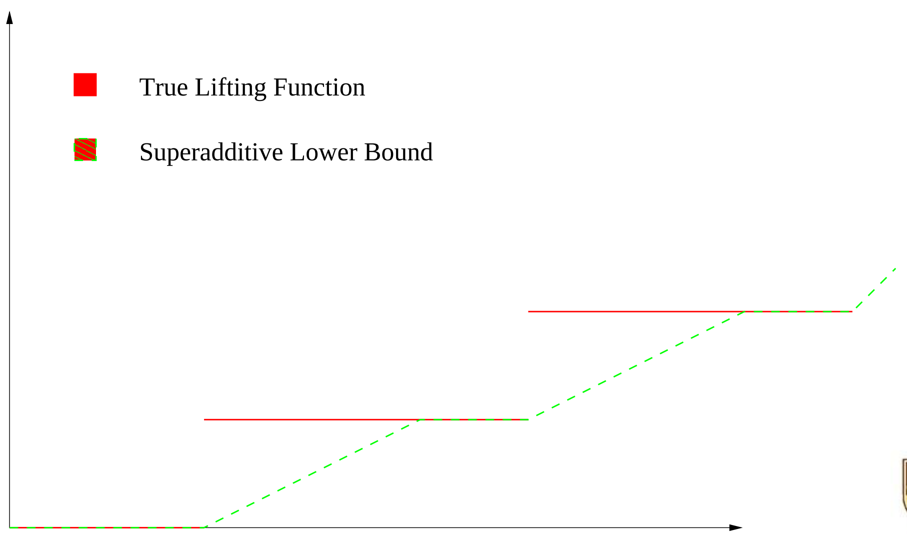
$$P = \text{conv}(\{x \in \mathbb{B}^{10} \mid 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39\})$$

- $C = \{4, 5, 6\}$, so $\lambda = 10$

$$\Theta(\alpha) = 20 - \max\{10x_4 + 10x_5 + 10x_6 \mid 35x_1 + 27x_2 + 23x_3 + 19x + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39 - \alpha\}$$

- AMPL Example...

$\Theta(\alpha)$



Superadditive?

- Is $\Theta(\alpha)$ superadditive?
  - **No!** $\alpha_1 = 10, \alpha_2 = 25$

$$\phi(\alpha) = \begin{cases} 0 & \text{if } 0 \leq \alpha \leq 9 \\ 10 + \alpha - 19 & \text{if } 9 \leq \alpha \leq 19 \\ 10 & \text{if } 19 \leq \alpha \leq 24 \\ 20 + \alpha - 34 & \text{if } 24 \leq \alpha \leq 34 \\ 20 & \text{if } 34 \leq \alpha \leq 39 \\ 30 + \alpha - 49 & \text{if } \alpha \geq 39 \end{cases}$$

- Using ϕ we get an inequality
  - $2x_1 + \frac{13}{10}x_2 + x_3 + x_4 + x_5 + x_6 + \frac{3}{10}x_7 \leq 2$

How Strong?

- We know that the cover inequality $x_3 + x_4 + x_5 \leq 2$ defines a facet of the restriced problem.
- Is our inequality a facet of K_2?
  - Does $\phi(a_i) = \Phi(a_i)\ \forall i \in L$ and $\phi(-a_i) = \Phi(-a_i)\ \forall i \in U$?
- **No!**. "Closest" facet is

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq 2$$