IE418: Integer Programming

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General Lifting and SuperAdditivity

- $K = \text{conv}(\{x \in \mathbb{Z}_+^{|N|}, y \in \Re_+^{|M|} \mid a^T x + g^T y \leq b, x \leq u\})$
- Partition N into $[L, U, R]$
  - $L = \{i \in N \mid x_i = 0\}$
  - $U = \{i \in N \mid x_i = u_i\}$
  - $R = N \setminus L \setminus U$
- We will use the notation: x_R to mean the vector of variables that are in the set R.
  - $a_R^T x_R = \sum_{j \in R} a_j x_j$

$$K(L,U) = \text{conv}(\{x \in \mathbb{Z}_+^{|N|}, y \in \Re_+^{|M|} \mid$$
$$a_R^T x + g^T y \leq d, x_R \leq u_R, x_i = 0 \ \forall i \in L, x_i = u_i \ \forall i \in U.\})$$

- So $d = b - a_U^T x_U$

Lifting

- Let $\pi^T x_R - \sigma^T y \leq \pi_0$ be a valid inequality for $K(L, U)$.
- Consider the *lifting function* $\Phi : \Re \rightarrow \Re \cup \{\infty\}$

$$\Phi(\alpha) = \pi_0 - \max\{\pi_R^T x_R + \sigma^T y \mid$$
$$a_R^T x_R + g^T y \leq d - \alpha, x_R \leq u_R, x_R \in \mathbb{Z}_+^{|R|}, y \in \Re_+^{|M|}\}$$

- (∞) if lifting problem is infeasible
- In words, $\Phi(\alpha)$ is the maximum value of the LHS of the valid inequality if the RHS in K is reduced by α.

Φ, Schmi

- Why do we care about Φ?

$$\pi_R^T x_R + \pi_L^T x_L + \pi_U^T (u_U - x_U) + \sigma^T y \leq \pi_0$$

is a valid inequality for K if and only if

$$\pi_L^T x_L + \pi_U^T (u_U - x_U) \leq \Phi(a_L^T x_L + a_U^T (x_U - u_U)) \ \forall (x, y) \in K.$$

Proof.

$$\Phi(a_L^T x_L + a_U^T (x_U - u_U)) = \pi_0 - \max\{\pi_R^T x_R + \sigma^T y \mid$$
$$a_R^T x_R + g^T y \leq b - a_L x_L - a_U x_U,$$
$$x_R \leq u_R, x_R \in \mathbb{Z}_+^{|R|}, y \in \mathbb{R}_+^{|M|}\}$$

So if there exists $(\hat{x}, \hat{y})$ such that $\pi_L^T \hat{x}_L + \pi_U^T (u_U - \hat{x}_U) + \max\{\} > \pi_0$, then $\pi_L^T \hat{x}_L + \pi_U^T (u_U - \hat{x}_U) + \pi_R^T x_R + \sigma^T y \leq \pi_0$ cannot be a valid inequality.

Sequential Lifting. Example

- Suppose that we are doing sequential lifting for $0-1$ IP like we have done so far.
- If x_k fixed at 0. (Lower bound). $\alpha x_k + \pi_R^T x_R \leq \pi_0$ is valid for $P \Leftrightarrow \alpha x_k \leq \Phi(a_k x_k)\ \forall x \in P$
  - $x_k = 0, \quad 0 \leq \Phi(0)$ is always true.
  - $x_k = 1, \quad \Rightarrow \alpha \leq \Phi(a_k)$
- If x_k fixed at one (Upper Bound), then $\alpha(1-x_k) + \pi_R^T x_R \leq \pi_0$ is valid for $P \Leftrightarrow \alpha(1-x_k) \leq \Phi(a_k(x_k-1)) \forall x \in P$
  - $x_k = 1, \quad 0 \leq \Phi(0)$ is always true.
  - $x_k = 0, \quad \Rightarrow \alpha \leq \Phi(-a_k)$
- For some classes of inequalities, we have closed form solution for the lifting function.
- If I "know" $\Phi(q)(\forall q \in \Re)$, I can just "lookup" the value of the lifting coefficient for variable x_k

Lifting Functions (Sequential)

- Note that if I have restricted more than one variable, then this "lookup" logic is not necessarily true
  - For lifting two (0-1) variables, I would have to look at four possible values.
- In general, the lifting function Φ for some valid inequality $\pi_R^T x_R + g^T y \leq \pi_0$ changes as I lift variables: $\Phi_{i+1}(\alpha) \neq \Phi_i(\alpha)\ \forall i, \alpha$
- This implies that if I lift the variables in different orders, I can get different facets.
- What do we know about relationships between lifting functions?
- It is monotonically decreasing: $\Phi_{i+1}(\alpha) \leq \Phi_i(\alpha) \forall i, \alpha$. (**Why?**—N&W II.2, Proposition 1.3)
- The highest value a coefficient can have when I lift it comes when I lift it first.

Lifting Functions

- Suppose the lifting function *doesn't change* when I lift a variable.
- If this happens, I can use the same lifting function again to determine the next coefficient.
- If the lifting function *never changes*, then I can use the same function to lift all of the variables.
- This happens *if and only if* Φ is a superadditive

Superadditivity

- A function $\phi : \Re \rightarrow \Re$ is *superadditive* if

$$\phi(q_1) + \phi(q_2) \leq \phi(q_1 + q_2)$$

- Superadditive functions play a significant role in the theory of integer programming. (See N&W page 229).
- Example: $\lfloor \cdot \rfloor$ is a superadditive function.
- Superadditive Fact:

$$\sum_{j \in N} \phi(a_j) x_j \leq \sum_{j \in N} \phi(a_j x_j) \leq \phi\left(\sum_{j \in N} a_j x_j\right).$$

"Multiple Lookup"—Superadditivity

- Suppose that ϕ is a superadditive lower bound on Φ that satisfies $\pi_i = \phi(a_i)\ \forall i \in L$ and $\pi_i = \phi(-a_i)\ \forall i \in U$

$$\begin{aligned}\sum_{i \in L} \phi(a_i) x_i + \sum_{i \in U} \phi(-a_i)(u_i - x_i) &\leq \phi(a_L^T x_L + a_U^T (x_U - u_U)) \\ &\leq \Phi(a_L^T x_L + a_U^T (x_U - u_U))\end{aligned}$$

- So
$$\pi_R^T x_R + \pi_L^T x_L + \pi_U^T (u_U - x_U) + \sigma^T y \leq \pi_0$$
is a valid inequality for K

The Main Result

- If ϕ is a superadditive lower bound on Φ, any inequality of the form $\pi_R^T x_R - \sigma^T y \leq \pi_0$, which is valid for $K(L, U)$, can be extended to the inequality
$$\pi_R^T x_R + \sum_{j \in L} \phi(a_j) x_j + \sum_{j \in U} \phi(-a_j)(u_j - x_j) + \sigma^T y \leq \pi_0$$
which is valid for K.
- If $\phi(a_i) = \Phi(a_i)\ \forall i \in L$ and $\phi(-a_i) = \Phi(a_i)\ \forall i \in U$ and $\pi^T x_R - \sigma^T y = \pi_0$ defines a k-dimensional face of $K(L, U)$, then the lifted inequality defines a face of dimension at least $k + |L| + |U|$ of K

This Is Sooooooooo Cool

- What does this imply?
- If the lifting function itself is superadditive, I can lift *all* of the variables in one pass (if I know the lifting function, of course).
- Even if I don't know the lifting function, if I can get a superadditive function that is a lower bound, then I can lift all the variables at once.
- Often, by examining the special structure of the lifting problem, one can fairly easily deduce a (closed form) solution for the lifting function.
- Then one can also deduce a superadditive lower bound

Example—Lifted Knapsack Covers

$$P = \text{conv}(\{x \in \mathbb{B}^{10} \mid 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39\})$$

- $C = \{4, 5, 6\}$, so $\lambda = 10$

$$\Theta(\alpha) = 20 - \max\{10x_4 + 10x_5 + 10x_6 \mid 35x_1 + 27x_2 + 23x_3 + 19x_4 + 15x_5 + 15x_6 + 12x_7 + 8x_8 + 6x_9 + 3x_{10} \leq 39 - \alpha\}$$

Superadditive?

- Is $\Theta(\alpha)$ superadditive?
  - **No!** $\alpha_1 = 10, \alpha_2 = 25$

$$\phi(\alpha) = \begin{cases} 0 & \text{if } 0 \leq \alpha \leq 9 \\ 10 + \alpha - 19 & \text{if } 9 \leq \alpha \leq 19 \\ 10 & \text{if } 19 \leq \alpha \leq 24 \\ 20 + \alpha - 34 & \text{if } 24 \leq \alpha \leq 34 \\ 20 & \text{if } 34 \leq \alpha \leq 39 \\ 30 + \alpha - 49 & \text{if } \alpha \geq 39 \end{cases}$$

- Using ϕ we get an inequality
  - $2x_1 + \frac{13}{10}x_2 + x_3 + x_4 + x_5 + x_6 + \frac{3}{10}x_7 \leq 2$

Facets of P

```
( 12) +  x1                          + x9     <=  1
( 13) +  x1                     + x8          <=  1
( 14) +  x1                + x7               <=  1
( 15) +  x1+ x2       + x6                    <=  1
( 16) +  x1+ x2  + x5                         <=  1
( 17) +  x1+ x2+ x3+ x4                       <=  1
( 18) + 2x1+ x2                 + x8+ x9      <=  2
( 19) +  x1+ x2            + x7          +x10 <=  2
( 20) + 2x1+ x2+ x3        + x7      + x9     <=  2
( 21) + 2x1+ x2+ x3        + x7+ x8           <=  2
( 22) + 2x1+2x2+ x3+ x4+ x5+ x6               <=  2
( 23) +  x1+ x2+ x3   + x6               +x10 <=  2
( 24) +  x1+ x2+ x3  + x5                +x10 <=  2
( 25) + 2x1+ x2+ x3+ x4   + x6       + x9     <=  2
( 26) + 2x1+ x2+ x3+ x4   + x6  + x8          <=  2
( 27) + 2x1+ x2+ x3+ x4+ x5          + x9     <=  2
( 28) + 2x1+ x2+ x3+ x4+ x5     + x8          <=  2
( 29) + 2x1+ x2+ x3+ x4+ x5+ x6+ x7           <=  2
( 30) + 3x1+2x2+2x3+2x4+ x5+ x6      + x9     <=  3
( 31) + 3x1+2x2+2x3+2x4+ x5+ x6 + x8          <=  3
( 32) + 2x1+2x2+2x3+ x4+ x5+ x6          +x10 <=  3
( 33) + 2x1+ x2+ x3             + x8+ x9+x10  <=  3
( 34) + 3x1+2x2+2x3+ x4+ x5+ x6+ x7  + x9     <=  3
( 35) + 3x1+2x2+2x3+ x4+ x5+ x6+ x7+ x8       <=  3
( 36) + 2x1+ x2+ x3+ x4        + x7    + x9+x10 <=  3
( 37) + 2x1+ x2+ x3+ x4        + x7+ x8    +x10 <=  3
( 38) + 2x1+2x2+ x3+ x4+ x5+ x6+ x7        +x10 <=  3
( 39) + 2x1+ x2+ x3+ x4+ x5+ x6   + x8     +x10 <=  3
( 40) + 3x1+2x2+ x3+ x4+ x5+ x6+ x7+ x8+ x9     <=  3
( 41) + 3x1+2x2+ x3+ x4       +2x7     + x9+x10 <=  4
( 42) + 3x1+2x2+ x3+ x4       +2x7+ x8     +x10 <=  4
( 43) + 3x1+3x2+2x3+ x4+ x5+2x6+ x7        +x10 <=  4
( 44) + 3x1+3x2+2x3+ x4+2x5+ x6+ x7        +x10 <=  4
( 45) + 3x1+2x2+2x3+ x4+ x5+2x6   + x8     +x10 <=  4
( 46) + 3x1+2x2+2x3+ x4+2x5+ x6   + x8     +x10 <=  4
( 47) + 4x1+3x2+2x3+2x4+ x5+2x6+ x7+ x8+ x9     <=  4
( 48) + 4x1+3x2+2x3+2x4+2x5+ x6+ x7+ x8+ x9     <=  4
( 49) + 4x1+2x2+2x3+ x4+ x5+ x6+2x7+ x8+ x9     <=  4
( 50) + 3x1+2x2+2x3+2x4+ x5+ x6+ x7    + x9+x10 <=  4
( 51) + 3x1+2x2+2x3+2x4+ x5+ x6+ x7+ x8    +x10 <=  4
( 52) + 3x1+2x2+2x3+ x4+ x5+ x6+ x7+ x8+ x9+x10 <=  4
( 53) + 4x1+4x2+3x3+2x4+2x5+2x6+ x7        +x10 <=  5
( 54) + 5x1+3x2+3x3+2x4+2x5+2x6+2x7+ x8+ x9     <=  5
( 55) + 5x1+4x2+3x3+3x4+2x5+2x6+ x7+ x8+ x9     <=  5
( 56) + 4x1+3x2+3x3+2x4+2x5+2x6+ x7+ x8    +x10 <=  5
( 57) + 4x1+3x2+3x3+2x4+ x5+2x6+ x7+ x8+ x9+x10 <=  5
( 58) + 4x1+3x2+3x3+2x4+2x5+ x6+ x7+ x8+ x9+x10 <=  5
( 59) + 4x1+3x2+2x3+2x4+ x5+ x6+2x7+ x8+ x9+x10 <=  5
```


But Wait There's More

```
( 60) + 5x1+3x2+3x3+3x4+2x5+2x6+ x7+2x8    +x10 <=  6
( 61) + 5x1+4x2+3x3+3x4+2x5+2x6+2x7+ x8+ x9+x10 <=  6
( 62) + 5x1+3x2+3x3+2x4+2x5+2x6+ x7+2x8+ x9+x10 <=  6
( 63) + 5x1+4x2+4x3+3x4+2x5+2x6+ x7+ x8+ x9+x10 <=  6
( 64) + 5x1+3x2+3x3+2x4+ x5+ x6+2x7+ x8+2x9+x10 <=  6
( 65) + 5x1+3x2+3x3+2x4+ x5+ x6+2x7+2x8+ x9+x10 <=  6
( 66) + 6x1+5x2+4x3+3x4+3x5+3x6+2x7+ x8    +x10 <=  7
( 67) + 6x1+4x2+4x3+3x4+2x5+2x6+2x7+ x8+2x9+x10 <=  7
( 68) + 6x1+4x2+4x3+3x4+2x5+3x6+ x7+2x8+ x9+x10 <=  7
( 69) + 6x1+4x2+4x3+3x4+3x5+2x6+ x7+2x8+ x9+x10 <=  7
( 70) + 7x1+5x2+4x3+3x4+2x5+2x6+3x7+2x8+2x9+x10 <=  8
( 71) + 7x1+5x2+5x3+4x4+3x5+3x6+2x7+2x8+ x9+x10 <=  8
( 72) + 7x1+5x2+5x3+4x4+2x5+3x6+2x7+ x8+2x9+x10 <=  8
( 73) + 7x1+5x2+5x3+4x4+3x5+2x6+2x7+ x8+2x9+x10 <=  8
( 74) + 8x1+6x2+5x3+4x4+3x5+3x6+3x7+2x8+2x9+x10 <=  9
( 75) + 8x1+6x2+6x3+5x4+3x5+3x6+2x7+ x8+2x9+x10 <=  9
( 76) + 9x1+7x2+6x3+5x4+3x5+4x6+3x7+2x8+2x9+x10 <= 10
( 77) + 9x1+7x2+6x3+5x4+4x5+3x6+3x7+2x8+2x9+x10 <= 10
( 78) + 9x1+7x2+6x3+5x4+4x5+4x6+3x7+2x8+ x9+x10 <= 10
( 79) +10x1+8x2+7x3+6x4+4x5+4x6+3x7+2x8+2x9+x10 <= 11
( 80) +12x1+9x2+8x3+6x4+5x5+5x6+4x7+3x8+2x9+x10 <= 13
```


What to do?

- We often try to solve problems that have knapsack rows with *lots* more variables than that...
- Obviously I do not want to add all of those facets.
- What to do?
- Given some $\hat{x} \notin P$, find an inequality of the form $\sum_{j \in C} x_j \leq |C| - 1$ such that $\sum_{j \in C} \hat{x}_j > |C| - 1$.
- This is called a *separation problem*
- Note that it is dependent on the particular class of inequalities—In this case cover inequalities.

Knapsack Separation

- Note that $\sum_{j \in C} x_j \leq |C| - 1$ can be rewritten as

$$\sum_{j \in C} (1 - x_j) \geq 1.$$

- Separation Problem: Given a "fractional" LP solution $\hat{x}$, does $\exists\ C \subseteq N$ such that $\sum_{j \in C} a_j > b$ and $\sum_{j \in C} (1 - \hat{x}_j) < 1$?
- Is $\gamma = \min_{C \subseteq N} \{\sum_{j \in C} (1 - \hat{x}_j) \mid \sum_{j \in C} a_j > b\} < 1$
- Let $z_j \in \{0, 1\}$, $z_j = 1$ if $j \in C$, $z_j = 0$ if $j \notin C$.
- Is $\gamma = \min\{\sum_{j \in N} (1 - \hat{x}_j) z_j \mid \sum_{j \in N} a_j z_j > b, z \in \mathbb{B}^n\} < 1$?
- If $\gamma \geq 1, \hat{x}$ satisfies all cover inequalities
- If $\gamma < 1$ with optimal solution z_R, then $\sum_{j \in R} x_j \leq |R| - 1$ is a violated cover inequality.

Example

$$\text{MYKNAP} = \{x \in \mathbb{B}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$$

- $\hat{x} = (0, 2/3, 0, 1, 1, 1, 1)$

$$\gamma = \min_{z \in \mathbb{B}^7} \{z_1 + 1/3 z_2 + z_3 \mid 11z_1 + 6z_2 + 6z_3 + 5z_4 + 5z_5 + 4z_6 + z_7 \geq 20\}.$$

- $\gamma = 1/3$
- $z = (0, 1, 0, 1, 1, 1, 1)$
- $x_2 + x_4 + x_5 + x_6 + x_7 \leq 4$
- Minimal Cover: $x_2 + x_4 + x_5 + x_6 \leq 3$
- You would do the lifting from here.

Complexity of Separation

- How hard is it to separate a fractional LP solution?
- Is it obvious that it is hard?
  - **No!** Since the point you are trying to separate is not an "arbitrary" knapsack problem, but instead the profits have a special form.
  - Klabjan, Nemhauser, and Tovey, "The Complexity of Cover Inequality Separation" show that knapsack separation is *NP*-Hard