IE418: Integer Programming

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Matching

- Let's Consider a New Graph Problem – Matching.
- Given a graph $G = (V, E)$ with weights on the edges $w_e \ \forall e \in E$, we are interested in finding a set of edges of maximum weight such that no two edges are incident on the same vertex.
- $\max_{x \in \mathbb{B}^{|E|}} \{\sum_{e \in E} w_e x_e \mid \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V\}$.
- Consider any set of nodes $T \subseteq V$ and add the "not more than one edge incident upon a vertex" constraint for these nodes.
  - If $e \in E(T)$, then we will count that edge twice
  - If $e \in \delta(T, V \setminus T)$, then we count that edge once
  - If $e \in E(V \setminus T)$, then we count this edge zero times

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Aggregated Inequality

- $2\sum_{e \in E(T)} x_e + \sum_{e \in \delta(T, V \setminus T)} x_e \leq |T|$
- $2\sum_{e \in E(T)} x_e \leq |T|$
- $\sum_{e \in E(T)} x_e \leq |T|/2$
- Suppose $|T|$ is odd, so $|T|/2 \notin \mathbb{Z}$
- $\sum_{e \in E(T)} x_e \leq \lfloor |T|/2 \rfloor$ is a valid inequality
- So What? What's the Magic Here?

It Is Magic!

$$X_1 = \text{conv}(\{x \in \mathbb{Z}_+^{|E|} \mid \sum_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V\})$$

$$\begin{aligned} X_2 = \{x \in \Re_+^{|E|} \mid \sum_{e \in \delta(v)} x_e &\leq 1 \ \forall v \in V, \\ \sum_{e \in E(T)} x_e &\leq (|T| - 1)/2 \ \forall T \subseteq V, |T| = 3, 5, \ldots, \} \end{aligned}$$

- Edmonds' Matching Polytope Theorme
  - $X_1 = X_2$
- The convex hull of matching is described by the degree constraints and the odd-set constraints
- Can we separate over the odd set constraints in polynomial time?
  - If so, then we can solve the weighted matching problem in polynomial time? (**How?**)

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Questions?

- Questions on Homework?
- Banquet? Who is attending?
- Final?

- Topics:
  - Aggregation and Rounding
  - Lagrangian Relaxation
  - Branch-and-price?
  - Preprocessing and Probing
  - Disjunctive Cuts?
  - IP Duality?

The Chvátal-Gomory Procedure

- Let the columns of $A \in \Re^{m \times n}$ be denoted by $\{a_1, a_2, \ldots a_n\}$
- $S = \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\}$.
  1. Choose nonnegative multipliers $u \in \Re^m_+$
  2. $u^T Ax \leq u^T b$ is a valid inequality ($\sum_{j \in N} u^T a_j x_j \leq u^T b$).
  3. $\sum_{j \in N} \lfloor u^T a_j \rfloor x_j \leq u^T b$ (Since $x \geq 0$).
  4. $\sum_{j \in N} \lfloor u^T a_j \rfloor x_j \leq \lfloor u^T b \rfloor$ is valid for X since $\lfloor u^T a_j \rfloor x_j$ is an integer

The Amazing Fact!

- The extremely simple logic/procedure described above is sufficient to generate *all* valid inequalities for an integer program.
- **Thm.** Every valid inequality for S can be obtained by applying the Chvátal-Gomory procedure a finite number of times.
  - The number of times that the procedure must be performed to obtain a certin inequality is called the *Chvátal-Gomory* rank of the inequality.
  - Thus, the odd-set inequalities for the matching polytope are rank-1 C-G inequalities.

Gomory's Cutting Plane Procedure for (Pure) IP

- $\max\{x \in \mathbb{Z}_+^n \mid Ax = b\}$
- Create the cutting planes directly from the simplex tableau
- Given an (optimal) LP basis B, write the (pure) IP as

$$\max c_B B^{-1} b + \sum_{j \in NB} \bar{c}_j x_j$$

$$\begin{aligned} x_{B_i} + \sum_{j \in NB} \bar{a}_{ij} x_j &= \bar{b}_i \ \forall i = 1, 2, \ldots m \\ x_j &\in \mathbb{Z} \ \forall j = 1, 2, \ldots n \end{aligned}$$

- NB is the set of nonbasic variables
- $\bar{c}_j \leq 0 \ \forall j$
- $\bar{b}_i \geq 0 \ \forall i$

Gomory's Cutting Planes

- If the LP solution is not integral, then there exists some row i with $\bar{b}_i \notin \mathbb{Z}$
- The C-G cut for row i is

$$x_{B_i} + \sum_{j \in NB} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor.$$

- Substitute for x_{B_i} to get

$$\sum_{j \in NB} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$$

- Or if $f_{ij} = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$, $f_i = \bar{b}_i - \lfloor \bar{b}_i \rfloor$, then

$$\sum_{j \in NB} f_{ij} x_j \geq f_i.$$

- Note that since $\hat{x}_j = 0 \ \forall j \in NB$ and x_{B_i} is fractional, then this is really a cut!

Example

$$\max 4x_1 - x_2$$

subject to

$$\begin{aligned} 7x_1 - 2x_2 &\leq 14 \\ x_2 &\leq 3 \\ 2x_1 - 2x_2 &\leq 3 \\ x_1, x_2 &\in \mathbb{Z}_+ \end{aligned}$$

Optimal Simplex Tableau

$$\max \frac{59}{7} - \frac{4}{7}x_3 - \frac{1}{7}x_4$$

$$\begin{aligned} x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 &= \frac{20}{7} \\ x_2 + x_4 &= 3 \\ -\frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 &= \frac{23}{7} \\ x_1, x_2, x_3, x_4, x_5 &\in \mathbb{Z}_+ \end{aligned}$$

- Cut from the first row of the tableau is

$$\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$$

or

$$x_6 = -\frac{6}{7} + \frac{1}{7}x_3 + \frac{2}{7}x_4$$

with $x_6 \in \mathbb{Z}_+$

Reoptimizing

$$\max \frac{15}{2} - \frac{1}{2}x_5 - 3x_6$$

$$\begin{aligned} x_1 + x_6 &= 2 \\ x_2 - \frac{1}{2}x_5 + x_6 &= \frac{1}{2} \\ x_3 - x_5 - 5x_6 &= 1 \\ x_4 + \frac{1}{2}x_5 + 6x_6 &= \frac{5}{2} \\ x_1, x_2, x_3, x_4, x_5, x_6 &\in \mathbb{Z}_+ \end{aligned}$$

- Cut from second row of tableau (in which x_2 is fractional) is

$$\frac{1}{2}x_5 \geq \frac{1}{2}$$

or

$$-\frac{1}{2}x_5 + x_7 = -\frac{1}{2}$$

Reoptimizing

$$\max 7 - 3x_6 - x_7$$

$$\begin{aligned} x_1 + x_6 &= 2 \\ x_2 + x_6 - x_7 &= 1 \\ x_3 - 5x_6 - 2x_7 &= 2 \\ x_4 + 6x_6 + x_7 &= 2 \\ x_5 - x_7 &= 1 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 &\in \mathbb{Z}_+ \end{aligned}$$

- **Done!**

Extension to Mixed Integer Programs

- One can show that the *Gomory Mixed Integer Cut* is a valid inequality for MIP.
- Given row of (mixed) tableau – **(y's are integers)**

$$T = \{(y_{B_i}, y, x) \in \mathbb{Z} \times \mathbb{Z}^{|N_1|} \times \mathbb{R}_+^{|N_2|} \mid y_{B_i} + \sum_{j \in N_1} \bar{a}_{ij} y_j + \sum_{j \in N_2} \bar{a}_{ij} x_j = \bar{b}_i\}$$

- Let $f_0 = \bar{b}_i - \lfloor \bar{b}_i \rfloor$, $f_j = \bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor$

$$\sum_{j \in N : f_j \leq f_0} f_j y_j + \sum_{j \in N : f_j > f_0} \frac{f_0(1 - f_j)}{1 - f_0} y_j + \sum_{j \in N : \bar{a}_{ij} > 0} \bar{a}_{ij} x_j + \sum_{j \in N : \bar{a}_{ij} < 0} \frac{f_0}{1 - f_0} \bar{a}_{ij} x_j \geq f$$

- Won't derive this, since we will show it's validity as a special case of a **Mixed Integer Rounding** inequality.

Mixed Integer Rounding—MIR

- Almost everything comes from considering the following very simple set, and very simple observation.
- Here, I will switch notation to use y as the integer valued variable, since that is what Marchand & Wolsey do.
- $X = \{(x, y) \in \mathbb{R} \times \mathbb{Z} \mid y \leq b + x\}$
- $y \leq \lfloor b \rfloor + \frac{1}{1-f} x$ is a valid inequality for X

(Simple) Extension of MIR

$$X = \{(x,y) \in \mathbb{R}^2_+ \times \mathbb{Z}^{|N|} \mid \sum_{j \in N} a_j y_j + x^+ \leq b + x^-\}$$

- $f = b - \lfloor b \rfloor$
- $f_j = a_j - \lfloor a_j \rfloor$
- The inequality

$$\sum_{j \in N} \left(\lfloor (a_j) \rfloor + \frac{(f_j - f)^+}{1-f} \right) y_j \leq \lfloor b \rfloor + \frac{x^-}{1-f}$$

  is valid for X
- X is a one-row relaxation of a general *mixed* integer program, where all of the continuous variables have been aggregated into two variables (one with positive coefficients), one with negative coefficients.

Proof.

- $N_1 = \{j \in N \mid f_j \leq f\}$
- $N_2 = N \setminus N_1$
- Let

$$P = \{(x,y) \in \mathbb{R}^2_+ \times \mathbb{Z}^{|N|} \mid \sum_{j \in N_1} \lfloor a_j \rfloor y_j + \sum_{j \in N_2} \lceil a_j \rceil y_y \leq b + x^- + \sum_{j \in N_2} (1 - f_j) y_j\}$$

1. Show $X \subseteq P$
2. Show Simple (2-variable) MIR inequality is valid for P (with an appropriate variable substitution.
3. Collect the terms

Proof 1. $X \subseteq P$

$$\begin{aligned}
(x,y) \in X \quad & \Rightarrow \quad \sum_{j \in N_1} a_j y_j + \sum_{j \in N_2} a_j y_j + x^+ \leq b + x^- \\
& \Rightarrow \quad \sum_{j \in N_1} \lfloor a_j \rfloor y_j + \sum_{j \in N_2} a_j y_j + x^+ \leq b + x^- \\
& \Rightarrow \quad \sum_{j \in N_1} \lfloor a_j \rfloor y_j + \sum_{j \in N_2} \lceil a_j \rceil y_j - \sum_{j \in N_2} (1 - f_j) y_j + x^+ \leq b + x^- \\
& \Rightarrow \quad (x,y) \in P
\end{aligned}$$

Proof 2.

- Let $w = \sum_{j \in N_1} \lfloor a_j \rfloor y_j + \sum_{j \in N_2} \lceil a_j \rceil y_j$. (Note that $w \in \mathbb{Z}$).
- Consider $w \leq b + x^- + \sum_{j \in N_2} (1 - f_j) y_j$
- Apply the "simple" MIR inequality to this set.

$$\sum_{j \in N_1} \lfloor a_j \rfloor y_j + \sum_{j \in N_2} \lceil a_j \rceil y_j \leq \lfloor b \rfloor + \frac{x^- + \sum_{j \in N_2} (1 - f_j) y_j}{1 - f}.$$

- This is an equivalent inequality to

$$\sum_{j \in N} (\lfloor (a_j) \rfloor + \frac{(f_j - f)^+}{1-f} y_j \leq \lfloor b \rfloor + \frac{x^-}{1-f}$$

Proof 3.

- Coefficient of y_j
  - $\lfloor a_j \rfloor$ if $j \in N_1$
  - $\lceil a_j \rceil - \frac{1-f_j}{1-f}$ if $j \in N_2$ (if $f_j > f$)

Gomory Mixed Integer Cut is a MIR Inequality

- Consider the set

$$X^{=} = \{(x, y_0, y) \in \mathbb{R}^2_+ \times \mathbb{Z} \times \mathbb{Z}^{|N|}_+ \mid y_0 + \sum_{j \in N} a_j y_j + x^+ - x^- = b\}$$

  which is essentially the row of an LP tableau with y_0 the basic variable and x^+, x^- the sum of the continuous variables with positive and negative coefficients.
- Relax the equality to an inequality and apply MIR

Proof.

$$\begin{aligned}
y_0 + \sum_{j \in N} \lfloor \left(a_j \rfloor + \frac{(f_j - f)^+}{1-f} \right) y_j &\leq \lfloor b \rfloor + \frac{x^-}{1-f} \\
b - \sum_{j \in N} a_j y_j - x^+ + x^- + \sum_{j \in N} \left(\lfloor a_j \rfloor + \frac{(f_j - f)^+}{1-f} \right) y_j &\leq \lfloor b \rfloor + \frac{x^-}{1-f} \\
-b + \sum_{j \in N} a_j y_j + x^+ - x^- - \sum_{j \in N} \lfloor \left(a_j \rfloor + \frac{(f_j - f)^+}{1-f} \right) y_j &\geq -\lfloor b \rfloor - \frac{x^-}{1-f} \\
\sum_{j \in N} f_j y_j + x^+ - x^- - \sum_{j \in N} \frac{(f_j - f)^+}{1-f} y_j &\geq f - \frac{x^-}{1-f} \\
\sum_{j \in N} f_j y_j + x^+ + \frac{f}{1-f} x^- - \sum_{j \in N_2} \frac{f_j - f}{1-f} y_j &\geq f \\
\sum_{j \in N_1} f_j y_j + x^+ + \frac{f}{1-f} x^- + \sum_{j \in N_2} (f_j - \frac{f_j - f}{1-f}) y_j &\geq f
\end{aligned}$$

Paper on Web Site

- Lots of inequalities are special cases of this inequality: Network design problems, Mixed cover inequalities, weight inequalities, etc.
- There is *lots* of interesting work to do to try and develop a good implementation. In fact the paper describes ways to aggregate, substitute, complement, and separate in order to find good inequalities for general MIP

READ!

- [1]
- [2]

H. Marchand and L. Wolsey, *Aggregation and mixed integer rounding to solve MIPs*, Operations Research, 49 (2001), pp. 363–371.

M. W. P. Savelsbergh, *Preprocessing and probing techniques for mixed integer programming problems*, ORSA Journal on Computing, 6 (1994), pp. 445–454.