IE418: Integer Programming. Decomposition Techniques.

Jeff Linderoth

Department of Industrial and Systems Engineering
Lehigh University

20th April 2005

The Lagrangian Relaxation

- The problem (IP) for now:

$$z^* = \max_{x \in X} \{c^T x \mid Dx \le d\}$$

- $X = \{x \in \mathbb{Z}^n_+ \mid Ax \le b\}$
- The constraints defining X are "nice" in the sense that we can solve $\max_{x \in X}\{c^T x\}$ effectively.
- Maybe X is a network problem
- Maybe X is a knapsack problem
- Maybe X has an efficient combinatorial algorithm

Lagrangian Dual

- Consider the problem (LR(u)) (for $u \in \Re^m_+$)

$$z(u) = \max_{x \in X}\{c^T x + u^T(d - Dx)\}$$

  - x feasible to IP $\Rightarrow$ x feasible to LR(u).
  - x feasible to IP, $u \ge 0 \Rightarrow c^T x + u^T(d - Dx) \ge c^T x$
  - $z(u) \ge z^* \; \forall u \ge 0$

- Since $z(u)$ provides an upper bound $\forall u \ge 0$, for bound-based algorithms, we would like for it to provide as tight a bound as possible:

$$z_{LD} = \min_{u \ge 0} z(u)$$

Strength of Lagrangian Relaxation

- We'll assume that X is bounded, so that it contains a finite number of points $S = \{x^1, x^2, \ldots x^s\}$.

$$\begin{aligned} z_{LD} &= \min_{u \ge 0} z(u) \\ &= \min_{u \ge 0} \max_{x \in X}\{c^T x + u^T(d - Dx)\} \\ &= \min_{u \ge 0} \max_{s \in \{1,2,\ldots |S|\}} \{c^T x^s + u^T(d - Dx^s)\} \\ &= \min_{u \ge 0, \eta \in \Re} \{\eta \mid \eta \ge c^T x^s + u^T(d - Dx^s) \; \forall s \in S\} \end{aligned}$$

- Take the LP Dual of the last problem

Strength of Lagrangian Relaxation

$$z_{LD} = \max \sum_{s \in S} \lambda_s (c^T x^s)$$

subject to

$$\begin{aligned} \sum_{s \in S} \lambda_s &= 1 \\ \sum_{s \in S} \lambda_s (Dx^s - d) &\leq 0 \\ \lambda_s &\geq 0 \quad \forall s \in S \end{aligned}$$

Grouping Terms

$$z_{LD} = \max c^T \left(\sum_{s \in S} \lambda_s x^s \right)$$

subject to

$$\begin{aligned} \sum_{s \in S} \lambda_s &= 1 \\ D \left(\sum_{s \in S} \lambda_s x^s \right) &\leq d \\ \lambda_s &\geq 0 \quad \forall s \in S \end{aligned}$$

$$x = \sum_{s \in S} \lambda_s x^s, \sum_{s \in S} \lambda_s = 1, \lambda_s \geq 0 \ \forall s \in S$$

$$z_{LD} = \max\{c^T x \mid Dx \leq d, x \in \text{conv}(X)\}$$

Things We Learned

A Fundamental Concept

Solving the Lagrangian Dual is equivalent to finding a convex combination of points in X that also satisfy the complicating constraint $Dx \leq d$

- $z_{LD} = \min_{u \geq 0} z(u)$
- $z(u) = \max_{s \in \{1,2,\ldots|S|\}} \{c^T x^s + u^T (d - Dx^s)\}$
- $z(u)$ is the maximum of a number of a number of linear functions, it is therefore a piecewise linear convex function.
- You can solve these problems using the subgradient method
- For those of you in Stochastic Programming – this should look slightly familiar

Subgradient Algorithm

- The idea of the subgradient algorithm is to first choose a u, then evaluate $z(u)$ and get a direction of improvement.
- Here is a basic subgradient algorithm for solving LD:
  1. Choose initial Lagrange multipliers $u^0 \geq 0$ and set $t = 0$.
  2. Solve the Lagrangian subproblem $LR(u)$.
  3. Calculate the current violation of the complicating constraints $s = d - Dx$.
  4. Set $u^{t+1} \leftarrow u^t - \mu^t \frac{s}{\|s\|}$ where μ^t is the chosen step size.
  5. Set $t \leftarrow t + 1$ and go to step 2.
- This algorithm is guaranteed to converge to the optimal solution as long as $\{\mu^t\}_{t=0}^{\infty} \rightarrow 0$ and $\sum_{t=0}^{\infty} \mu^t = \infty$
- Convergence is slow

Comparing LP relaxation to LR

- $z_{IP} \stackrel{\text{def}}{=} \max\{c^T x \mid Ax \le b, Dx \le d, x \in \mathbb{Z}^n_+\}$
- $Dx \le d$ are the complicating constraints
- $X = \{x \in \mathbb{Z}^n_+ \mid Ax \le b\}$
- $\mathcal{D} = \{x \in \mathbb{Z}^n_+ \mid Dx \le d\}$
- $z_{LD} \stackrel{\text{def}}{=} \min_{u \ge 0} \max_{x \in X} \{c^T x + u^T(d - Dx)\}$
- $z_{IP} \stackrel{\text{def}}{=} \max\{c^T x \mid x \in \text{conv}(X \cap \mathcal{D})\}$
- $z_{LPCONV} \stackrel{\text{def}}{=} \max\{c^T x \quad Dx \le d, x \in \text{conv}(X)\}$

Our Key Theorem

$$z_{LD} = z_{LPCONV}$$

What About z_{LP}?

- $R(X) = \{x \in \mathbb{R}^n_+ \mid Ax \le b\}$
- $z_{LP} = \max\{c^T x \mid Dx \le d, x \in R(X)\}$
- $\text{conv}(X) \subseteq R(X)$
- $\Rightarrow z_{LP} \ge z_{LPCONV} = z_{LD}$

Two key points

- Bound obtained from solving Lagrangian Dual is sure to be at least as tight as that from solving the LP relaxation
- If $R(X) = \text{conv}(X)$, i.e. if X has all integer extreme points, then $z_{LP} = z_{LD}$: The bounds are the same!

Dantzig-Wolfe Decomposition

- A way to compute z_{LPCONV} directly.
- Assume that X is bounded (just for simplicity), with extreme points $T = \{p_1, p_2, \ldots p_{|T|}\}$
- $\text{conv}(X) = \{x \in \mathbb{R}^n \mid x = \sum_{t \in T} \lambda_t p_t, \sum_{t \in T} \lambda_t = 1, \lambda_t \ge 0 \ \forall t \in T\}$

$$z_{LPCONV} = \max c^T x$$

subject to

$$\begin{aligned} Dx &\le d \\ x &\in \text{conv}(X) \end{aligned}$$

$$z_{DW} \stackrel{\text{def}}{=} z_{LPCONV} = \max c^T \left(\sum_{t \in T} \lambda_t p_t\right)$$

subject to

$$\begin{aligned} D\left(\sum_{t \in T} \lambda_t p_t\right) &\le d \\ \sum_{t \in T} \lambda_t &= 1 \\ \lambda_t &\ge 0 \quad \forall t \in T \end{aligned}$$

Branch-and-Price

- [1]
- [2]

C. Barnhart, E. L. Johnson, G. L. Nemhauser, M. W. P. Savelsbergh, and P. H. Vance, *Branch and price: Column generation for solving huge integer programs*, Operations Research, 46 (1998), pp. 316–329.

F. Vanderbeck and M. Savelsbergh, *A generic view at the Dantzig-Wolfe decomposition approach in mixed integer programming*, Operations Research Letters, (2005). Submitted.

Benders' Decomposition

- Up until now, we have looked at the idea of complicating constraints.
- Benders' decomposition is based on the notion of complicating variables.
- Suppose we have the MIP
- Note that for a fixed x, this is a linear program.
- Consider the fixed-charge network flow problem: if the set of open arcs is fixed, the problem becomes easy.

$$\max c^T x + h^T y$$

subject to

$$\begin{aligned} Ax + Gy &\leq b \\ x &\in \mathbb{Z}_+^n \\ y &\in \mathbb{R}_+^p \end{aligned}$$

Formulating Benders' Decomposition

- First, assume x is fixed to obtain the resulting linear program

$$z_{LP}(x) = \max\{hy \mid Gy \leq b - Ax\}$$

and its dual

$$\min\{u(b - Ax) \mid uG \geq h, u \in \Re_+^m\}$$

- Assuming the dual polyhedron is nonempty and bounded, MIP can be restated as

$$z = \max_{x \in \mathbf{Z}_+^n} \left(cx + \min_{i \in 1,\ldots,T} u^i(b - Ax) \right)$$

where $\{u^i\}_{i=1}^T$ are the extreme points of the dual polyhedron.

Formulating Benders' Decomposition (cont.)

- As before, we can reformulate this as

$$z = \max\{\eta \mid \eta \leq u^i(b - Ax), i \in 1, \ldots, T, x \in \mathbf{Z}_+^n\}$$

- We can again in theory solve this formulation using constraint generation.
- The main use of this technique is when A is block decomposable and the resulting IP is much easier than the original.
- There are also various relaxations to be obtained from this formulation.