IE418: Integer Programming

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Review

- Name two reasons why anyone would want to "program with integers"?
- What is the knapsack problem?
- (MIP): $\max\{c^T x + h^T y \mid Ax + Gy \leq b, x \in \mathbb{Z}^n_+, y \in \mathbb{R}^p_+\}$
  - What is Mixed 0-1 Programming?
  - What is Pure Integer Programming
  - What is Binary Programming?
  - What is 0-1 Programming?
- What is $\mathbb{Q}^n_+$?
- What is my son's name?

Outline

- Combinatorial Optimization Problems
  - Set {Packing, Covering, Partitioning}
  - TSP
- Special Ordered Sets
  - Models of Choice—"Pick one!"
  - Piecewise Linear Functions
- Algorithimic Modeling
  - The Bag of Tricks
  - A couple examples

Selecting from a Set

- We can use constraints of the form $\sum_{j \in T} x_j \geq 1$ to represent that at least one item should be chosen from a set T.
  - Similarly, we can also model that at most one or exactly one item should be chosen.
- Example: Set covering problem
- If A in a 0-1 matrix, then a set covering problem is any problem of the form

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \geq e^1 \\ & x_j \in \{0,1\} \quad \forall j \end{aligned}$$

- Set Packing: $Ax \leq e$
- Set Partitioning: $Ax = e$

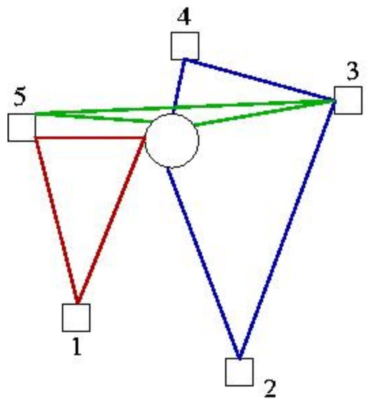
[1] It is common to denote the vector of 1's as e

COP—Set{Cover, Pack, Partition}ing

- A combinatorial optimization problem $CP = (N, \mathcal{F})$ consists of
  - A finite ground set N,
  - A set $\mathcal{F} \subseteq 2^N$ of feasible solutions, and
  - A cost function c

- Each row of A represents an item from N
- Each column A_j represents a subset $N_j \in \mathcal{F}$ of the items.
- Each variable x_j represents selecting subset N_j.
- The constraints (for covering and partitioning) say that $\cup_{\{j | x_j = 1\}} N_j = N$.
- In other words, each item must appear in at least, (or exactly one) selected subset.

Vehicle Routing



	x_1	x_2	x_3	$\ldots$		
Customer 1 :	1	0	0	$\vdots$	=	1
Customer 2 :	0	1	0	$\vdots$	=	1
Customer 3 :	0	1	1	$\vdots$	=	1
Customer 4 :	0	1	0	$\vdots$	=	1
Customer 5 :	1	0	1	$\vdots$	=	1

- This is a very flexible modeling trick
- You can list all feasible routes, allowing you to handle "weird" constraints like time windows, strange precedence rules, nonlinear cost functions, etc.

The Farmer's Daughter

- This is *The Most Famous Problem in Combinatorial Optimization!*
- A traveling salesman must visit all his cities at minimum cost.
- Given directed (complete) graph with node set N. $(G = (N, N \times N))$
- Given costs c_{ij} of traveling from city i to city j
- Find a minimum cost Hamiltonian Cycle in G
- Variables: $x_{ij} = 1$ if and only if salesman goes from city i to city j

TSP (cont.)

$$\min \sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}$$

$$\sum_{i \in N} x_{ij} = 1 \quad \forall j \in N \quad \text{Enter Each City}$$

$$\sum_{j \in N} x_{ij} = 1 \quad \forall i \in N \quad \text{Leave Each City}$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in N, \forall j \in N$$

Subtour elimination constraint: ("No Beaming")

$$\sum_{i \in S} \sum_{j \notin S} x_{ij} \geq 1 \quad \forall S \subseteq N, 2 \leq |S| \leq |N| - 2$$

Alternatively: ("No Beaming")

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1 \quad \forall S \subseteq N, 2 \leq |S| \leq |N| - 2$$

TSP Trivia Time!

What is This Number?

1018517988167243043134222844204689080525734196832968125318070224677190649881668353091698688.

- Is this...
  a) The number of times an undergraduate student asked me where room 355 was this week?
  b) The number of subatomic particles in the universe?
  c) The number of subtour elimination constraints when $|N| = 299$.
  d) All of the above?
  e) None of the above?

Answer Time

- The answer is (e). (a)–(c) are all too small (as far as I know) :-). (It is (c), for $|N| = 300$).
- "Exponential" is **really** big.
- Yet people have solved TSP's with $|N| > 16,000$!
- You will learn how to solve these problems too!
- The "trick" is to only add the subset of constraints that are necessary to prove optimality.
  - This is a trick known as branch-and-cut, and you will learn lots about branch-and-cut in this course.

Modeling a Restricted Set of Values

- We may want variable x to only take on values in the set $\{a_1, \ldots, a_m\}$.
- We introduce m binary variables $y_j, j = 1, \ldots, m$ and the constraints

$$x = \sum_{j=1}^{m} a_j y_j,$$
$$\sum_{j=1}^{m} y_j = 1,$$
$$y_j \in \{0, 1\}$$

- The set of variables $\{y_1, y_2, \ldots y_m\}$ is called a special ordered set (SOS) of variables.

Example—Building a warehouse

- Suppose we are modeling a facility location problem in which we must decide on the size of a warehouse to build.
- The choices of sizes and their associated cost are shown below:

Size	Cost
10	100
20	180
40	320
60	450
80	600

Warehouse sizes and costs

Warehouse Modeling

- Using binary decision variables $x_1, x_2, \ldots, x_5$, we can model the cost of building the warehouse as

$$\text{COST} \equiv 100x_1 + 180x_2 + 320x_3 + 450x_4 + 600x_5.$$

- The warehouse will have size

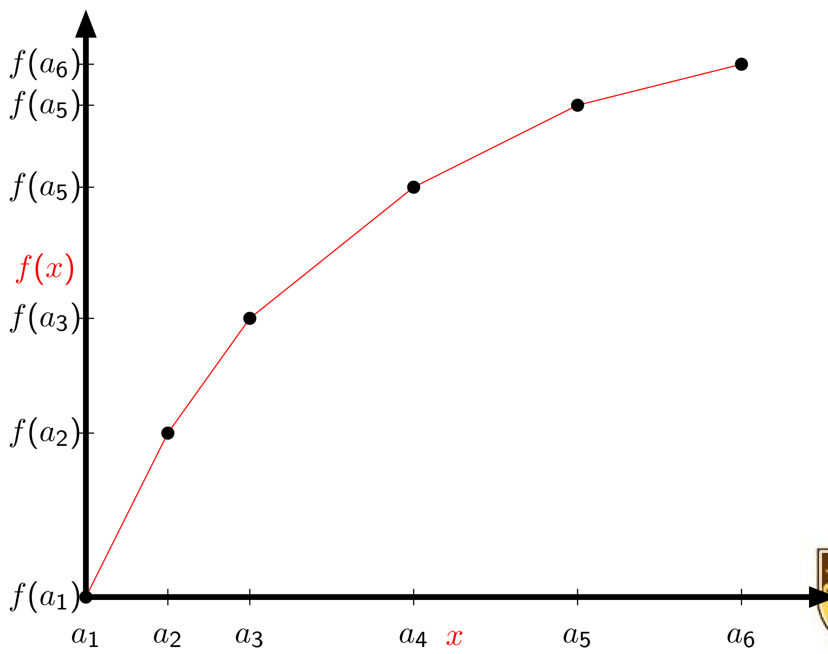
$$\text{SIZE} \equiv 10x_1 + 20x_2 + 40x_3 + 60x_4 + 80x_5,$$

- and we have the SOS constraint

$$x_1 + x_2 + x_3 + x_4 + x_5 = 1.$$

Piecewise Linear Cost Functions

- We can use binary variables to model arbitrary piecewise linear functions.
- The function is specified by ordered pairs $(a_i, f(a_i))$
- We have a binary variable y_i, which indicates whether $a_i \leq x \leq a_{i+1}$.



Minimizing Piecewise Linear Cost Functions

- To evaluate the function, we will take linear combinations $\sum_{i=1}^{k} \lambda_i f(a_i)$ of the given functions values.
- This only works if the only two nonzero $\lambda_i' s$ are the ones corresponding to the endpoints of the interval in which x lies.

The Key Idea!

If $y_j = 1$, then $\lambda_i = 0, \ \forall i \neq j, j{+}1$.

$$\min \sum_{i=1}^{k} \lambda_i f(a_i)$$

$$\begin{array}{rcl} \text{s.t.} \sum_{i=1}^{k} \lambda_i & = & 1, \\ \lambda_1 & \leq & y_1, \\ \lambda_i & \leq & y_{i-1} + y_i, \ i = 2, \ldots, k-1, \\ \lambda_k & \leq & y_{k-1}, \\ \sum_{i=1}^{k-1} y_i & = & 1, \\ \lambda_i & \geq & 0, \\ y_i & \in & \{0, 1\}. \end{array}$$

SOS2

- A "better" formulation involves the use of special ordered sets of type 2

SOS2

A set of variables of which at most two can be positive. If two are positive, they must be adjacent in the set.

$$\min \sum_{i=1}^{k} \lambda_i f(a_i)$$

$$\begin{array}{rcl} \text{s.t.} \sum_{i=1}^{k} \lambda_i & = & 1 \\ \lambda_i & \geq & 0 \\ \{\lambda_1, \lambda_2, \ldots, \lambda_k\} & & \text{SOS2} \end{array}$$

- The adjacency conditions of SOS2 are enforced by the solution algorithm
- (All) commercial solvers allow you to specify SOS2

Modeling Disjunctive Constraints

- We are given two constraints $a^T x \geq b$ and $c^T x \geq d$ with nonnegative coefficients.
- Instead of insisting both constraints be satisfied, we want at least one of the two constraints to be satisfied.
- To model this, we define a binary variable y and impose

$$\begin{aligned} a^T x &\geq yb, \\ c^T x &\geq (1-y)d, \\ y &\in \{0,1\}. \end{aligned}$$

- More generally, we can impose that at least k out of m constraints be satisfied with

$$\begin{aligned} (a_i')^T x &\geq b_i y_i, \quad i \in \{1,2,\ldots m\} \\ \sum_{i=1}^m y_i &\geq k, \\ y_i &\in \{0,1\} \end{aligned}$$

The Bag of Tricks

- There are lots of things you can model with binary variables:

- Indicator variables (Positivity of variables)
  - Limiting the Number of Positive Variables
  - "Fixed Charge" problems
  - Minimum production level
- Indicator variables (Validity of constraints)
  - Either-or
  - If-then
  - k out of n

The Bag of Tricks

- Special Ordered Sets
- Nonconvex regions
- Economies of Scale
- Discrete Capacity Extensions
- Maximax or Minimin

- The problem is that sometimes to see the modeling "trick" is difficult. For example...
- Use a 0-1 variable δ to indicate whether *or not* the constraint $2x_1 + 3x_2 \leq 1$ is satisfied.
  - x_1, x_2 are nonnegative continuous variables that are ≤ 1
- $\delta = 1 \Leftrightarrow 2x_1 + 3x_2 \leq 1$

The Slide of Tricks. Indicator Variables...

- $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b$
  - $\sum_{j \in N} a_j x_j + M\delta \leq M + b$
- $\sum_{j \in N} a_j x_j \leq b \Rightarrow \delta = 1$
  - $\sum_{j \in N} a_j x_j - (m - \epsilon)\delta \geq b + \epsilon$
- $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \geq b$
  - $\sum_{j \in N} a_j x_j + m\delta \geq m + b$
- $\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1$
  - $\sum_{j \in N} a_j x_j - (M + \epsilon)\delta \leq b - \epsilon$

Definitions

- δ: Indicator variable ($\delta \in \{0, 1\}$).
- M: Upper bound on $\sum_{j \in N} a_j x_j - b$
- m: Lower bound on $\sum_{j \in N} a_j x_j - b$
- ϵ: Small tolerance beyond which we regard the constraint as haven been broken.
  - If $a_j \in \mathbb{Z}$, $x_j \in \mathbb{Z}$, then we can take $\epsilon = 1$.

Modeling Trick #1

- Indicating Constraint (Non)violation
- Suppose we wish to indicate whether or not an inequality $\sum_{j \in N} a_j x_j \leq b$ holds by means of an indicator variable δ.

Implication We Wish to Model

$$\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b$$

- This can be represented by the constraint
  - $\sum_{j \in N} a_j x_j + M\delta \leq M + b$

Trick #1... The Logic

$$\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b \Leftrightarrow \sum_{j \in N} a_j x_j + M\delta \leq M + b$$

- (Thinking)...
  - $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j - b \leq 0$
  - $1 - \delta = 0 \Rightarrow \sum_{j \in N} a_j x_j - b \leq 0$
  - $\sum_{j \in N} a_j x_j - b \leq M(1 - \delta)$
- Does it work?
  - $\delta = 0 \Rightarrow \sum_{j \in N} a_j x_j - b \leq M$
    - (true by definition of M)
  - $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j - b \leq 0$

Modeling Trick #2

$$\boxed{\sum_{j \in N} a_j x_j \leq b \Rightarrow \delta = 1}$$

- $\delta = 0 \Rightarrow \sum_{j \in N} a_j x_j \not\leq b$
- $\delta = 0 \Rightarrow \sum_{j \in N} a_j x_j > b$
- $\delta = 0 \Rightarrow \sum_{j \in N} a_j x_j \geq b + \epsilon$
  - If a_j, x_j are integer, we can choose $\epsilon = 1$
- Model as $\sum_{j \in N} a_j x_j - (m - \epsilon)\delta \geq b + \epsilon$
  - m is a lower bound for the expression $\sum_{j \in N} a_j x_j - b$

Some Last Modeling Tricks

$$\boxed{\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \geq b}$$

- Model as $\sum_{j \in N} a_j x_j + m\delta \geq m + b$

$$\boxed{\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1}$$

- Model as $\sum_{j \in N} a_j x_j - (M + \epsilon)\delta \leq b - \epsilon$
- You can obtain these by just transforming the constraints to $\leq$ form and using the first two tricks.

Back To Our Example...

- Use a 0-1 variable δ to indicate whether *or not* the constraint $2x_1 + 3x_2 \leq 1$ is satisfied.
  - x_1, x_2 are nonnegative continuous variables that cannot exceed 1.
- $\delta = 1 \Leftrightarrow 2x_1 + 3x_2 \leq 1$
- M : Upper Bound on $2x_1 + 3x_2 - 1$. 4 works
- m : Lower Bound on $2x_1 + 3x_2 - 1$. -1 works.
- ϵ : 0.1

Review | Disjunctions
Combinatorial Optimization Problems | The Bag of Tricks
Special Ordered Sets | Example #1
"Algorithmic" Modeling | Example #2: PPP

Example, Cont.

- ($\Rightarrow$) Recall the trick.
  - $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \leq b \Leftrightarrow \sum_{j \in N} a_j x_j + M\delta \leq M + b$
- $2x_1 + 3x_2 + 4\delta \leq 5$
- ($\Leftarrow$). Recall the trick.
  - $\sum_{j \in N} a_j x_j \leq b \Rightarrow \delta = 1 \Leftrightarrow \sum_{j \in N} a_j x_j - (m - \epsilon)\delta \geq b + \epsilon$
- $2x_1 + 3x_2 + 1.1\delta \geq 1.1$

$$\begin{aligned} 2x_1 + 3x_2 + 4\delta &\leq 5 \\ 2x_1 + 3x_2 + 1.1\delta &\geq 1.1 \end{aligned}$$

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A More Realistic Example

- PPP—Production Planning Problem. (A simple linear program).
- An engineering plant can produce five types of products: $p_1, p_2, \ldots p_5$ by using two production processes: grinding and drilling. Each product requires the following number of hours of each process, and contributes the following amount (in hundreds of dollars) to the net total profit.

	p_1	p_2	p_3	p_4	p_5
Grinding	12	20	0	25	15
Drilling	10	8	16	0	0
Profit	55	60	35	40	20

PPP – More Info

- Each unit of each product take 20 manhours for final assembly.
- The factory has three grinding machines and two drilling machines.
- The factory works a six day week with two shifts of 8 hours/day. Eight workers are employed in assembly, each working one shift per day.

PPP

maximize

$$55x_1 + 60x_2 + 35x_3 + 40x_4 + 20x_5 \qquad \text{(Profit/week)}$$

subject to

$$\begin{array}{rcll}
12x_1 + 20x_2 + 0x_3 + 25x_4 + 15x_5 & \leq & 288 & \text{(Grinding)} \\
10x_1 + 8x_2 + 16x_3 + 0x_4 + 0x_5 & \leq & 192 & \text{(Drilling)} \\
20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 & \leq & 384 & \text{Final Assembly} \\
x_i & \geq & 0 & \forall i = 1, 2, \ldots 5
\end{array}$$

Another PPP Modeling Example

- Let's model the following situation.
  - If we manufacture $P1$ or $P2$ (or both), then at least one of $P3$, $P4$, $P5$ must also be manufactured.
- We first need an indicator variable z_j that indicate when each of the $x_j > 0$.
  - How do we model $x_j > 0 \Rightarrow z_j = 1$?.
  - Hint: This is equivalent to $z_j = 0 \Rightarrow x_j = 0$

Modeling the Logic

Answer: $x_j \leq M z_j$

- Given that we have included the constraints $x_j \leq M z_j$, we'd like to model the following implication:
  - $z_1 + z_2 \geq 1 \Rightarrow z_3 + z_4 + z_5 \geq 1$
- Can you just "see" the answer?
- I can't. So let's try our "formulaic" approach.
- Think of it in two steps
  - $z_1 + z_2 \geq 1 \Rightarrow \delta = 1$
  - $\delta = 1 \Rightarrow z_3 + z_4 + z_5 \geq 1$.

Look up the Tricks

- First we model the following:
  - $z_1 + z_2 \geq 1 \Rightarrow \delta = 1$
- The formula from the bag o' tricks
- $\sum_{j \in N} a_j x_j \geq b \Rightarrow \delta = 1 \Leftrightarrow \sum_{j \in N} a_j x_j - (M + \epsilon)\delta \leq b - \epsilon$
- M : Upper Bound on $\sum_{j \in N} a_j z_j - b$
  - $M = 1$ in this case. $(z_1 \leq 1, z_2 \leq 1, b = 1)$.
- ϵ : "Tolerance Level" indicating the minimum about by which the constraint can be violated.
  - $\epsilon = 1$ in this case!
  - If the constraint is going to be violated, then it will be violated by at least one.

Modeling $z_1 + z_2 \geq 1 \Rightarrow \delta = 1$, Cont.

- Just plug in the formula $\sum_{j \in N} a_j x_j - (M + \epsilon)\delta \leq b - \epsilon$
  - $z_1 + z_2 - 2\delta \leq 0$
- Does this do what we want?

z_1	z_2	δ
0	0	≥ 0
0	1	$\geq 1/2 (\Rightarrow = 1)$
1	0	$\geq 1/2 (\Rightarrow = 1)$
1	1	≥ 1

Second Part

- Want to model the following:
  - $\delta = 1 \Rightarrow z_3 + z_4 + z_5 \geq 1$.
- The formula from the bag o' tricks
- $\delta = 1 \Rightarrow \sum_{j \in N} a_j x_j \geq b \Leftrightarrow \sum_{j \in N} a_j x_j + m\delta \geq m + b$
- m : lower bound on $\sum_{j \in N} a_j x_j - b$.
  - $m = -1$. $(z_1 \geq 0, z_2 \geq 0, b = 1)$.
- Plug in the formula:
  - $z_3 + z_4 + z_5 - \delta \geq 0$
- It works! (Check for $\delta = 0, \delta = 1$).

PPP, Make 1 or 2 $\Rightarrow$ make 3, 4, or 5

maximize

$$55x_1 + 60x_2 + 35x_3 + 40x_4 + 20x_5 \qquad \text{(Profit/week)}$$

subject to

$$\begin{aligned}
12x_1 + 20x_2 + 0x_3 + 25x_4 + 15x_5 &\leq 288 \\
10x_1 + 8x_2 + 16x_3 + 0x_4 + 0x_5 &\leq 192 \\
20x_1 + 20x_2 + 20x_3 + 20x_4 + 20x_5 &\leq 384 \\
x_i &\leq M_i z_i \qquad \forall i = 1, 2, \ldots 5 \\
z_1 + z_2 - 2\delta &\leq 0 \\
z_3 + z_4 + z_5 - \delta &\geq 0 \\
x_i &\geq 0 \qquad \forall i = 1, 2, \ldots 5 \\
z_i &\in \{0, 1\} \forall i = 1, 2, \ldots 5 \\
\delta &\in \{0, 1\}
\end{aligned}$$

Cool Things You Can Now Do

- Either constraint 1 or constraint 2 must hold
  - Create indicators δ_1, δ_2, then $\delta_1 + \delta_2 \geq 1$
- At least one constraint of all the constraints in M should hold
  - $\sum_{i \in M} \delta_i \geq 1$
- At least k of the constraints in M must hold
  - $\sum_{i \in M} \delta_i \geq k$
- If x, then y
  - $\delta_y \geq \delta_x$

That's All Folks!

- That's it for modeling.
- You should have read N&W I.1.1 by now.
- Those of you who want to get a jump start on the homeworks can consider doing problems
  - ???
  - ???