IE 495 – Final Exam

Due Date: May 1, 2003

Do the following problems. You *must* work alone on this exam. The *only* reference materials you are allowed to use on the exam are the textbook by Birge and Louveaux, the Stochastic Programming book by Kall and Wallace:

`http://www.unizh.ch/ior/Pages/Deutsch/Mitglieder/Kall/bib/ka-wal-94.pdf`,

the class notes and materials on the course web page, and references to help you with AMPL syntax. You may use whatever software tools you find necessary (e.g. AMPL, Excel, Maple, NEOS). If you feel you need other reference materials to correctly answer any problem, contact Prof. Linderoth.

Please show your work. You may attach printouts of AMPL files, Maple worksheets, and Excel worksheets, or you may email them to Prof. Linderoth.

1 You Say To-MAY-To, I Say To-MAH-To

Linear Programming[1]

Teresa's Terrific Tomatoes (TTT) is in the business of selling a set P of tomato-related products (e.g salsa, ketchup, tomato paste). In order to create these products, resources from a set R are required (e.g. tomatoes, sugar, labor, spices). To be specific, the parameter a_{pr} is the amount of resource $r \in R$ that is required to produce one unit of product $p \in P$. There is a limit b_r on the amount of each resource $r \in R$ that can be used in a production period. The company can pay for extra resources at a cost of $\gamma_r, r \in R$.

TTT management is trying to determine the optimal way to meet an estimate of its demand d_{pt} for each of its products $p \in P$ in each planning period t over a horizon T. The regular production costs for each product are $c_{pt}, p \in P, t \in T$. Any surplus production of a product $p \in P$ must be stored at a cost of α_p per unit. TTT management also considers unmet demand important, so it imposes a penalty cost of β_p for each unit of unmet demand for product $p \in P$. TTT carries no initial inventory of products.

1.1 Problem

Formulate a linear programming model that will tell TTT management how to minimize the combination of production, resource, surplus, and unmet demand costs, subject to their resource constraints.

1.2 Problem

Solve your model in Problem 1.1 given the sets of products

- $P = \{$Salsa, Ketchup, Tomato-Paste$\}$.
- $R = \{$Tomatoes, Sugar, Labor, Spices$\}$.

and the instance data in Tables 1— 5.

[1] If you miss this, Ralphs is in trouble

p	r	a_{pr}
salsa	tomatoes	0.5
salsa	sugar	1.0
salsa	labor	1.0
salsa	spices	3.0
ketchup	tomatoes	0.5
ketchup	sugar	0.5
ketchup	labor	0.8
ketchup	spices	1.0
tomato paste	tomatoes	1.0
tomato paste	sugar	0
tomato paste	labor	0.5
tomato paste	spices	0.25

Table 1: Resource Requirements for TTT. (a_{pr})

r	b_r	g_r
Labor	200	2.0
Tomatoes	250	0.5
Sugar	300	1.0
Spices	100	1.0

Table 2: Resources and Per Unit Additional Resource Costs for TTT. (b_r, g_r)

p	t	c_{pt}
Tomato-Paste	1	1.0
Tomato-Paste	2	1.1
Tomato-Paste	3	1.2
Ketchup	1	1.5
Ketchup	2	1.75
Ketchup	3	2.0
Salsa	1	2.5
Salsa	2	2.75
Salsa	3	3

Table 3: Production Costs for TTT. (c_{pt})

p	α_p	β_p
Tomato-Paste	0.5	4.0
Ketchup	0.25	6.0
Salsa	0.2	12.0

Table 4: Inventory and Shortage Costs for TTT. (α_p, β_p)

p	t	d_{pt}
Tomato-Paste	1	100
Tomato-Paste	2	100
Tomato-Paste	3	200
Ketchup	1	30
Ketchup	2	30
Ketchup	3	40
Salsa	1	5
Salsa	2	5
Salsa	3	20

Table 5: Product Demands for TTT. (d_{pt})

Stochastic Programming[2]

Suppose now that the product demands are not known for certain until after the production decisions are made. To account for the uncertainty, TTT forecasts a number of demand scenarios $S = \{1, 2, \dots |S|\}$ where d_{pts} is the demand for product $p \in P$ in time period $t \in T$ under scenario $s \in S$. The probability of a scenario $s \in S$ occurring is ρ_s.

1.3 Problem
Referring to your answer to problem 1.1, which of the variables are *first-stage* variables, and which of the variables are *second-stage* variables.

1.4 Problem
Formulate a linear programming deterministic equivalent that will tell TTT management how to minimize their *expected* costs under varying demand scenarios.

1.5 Problem
Solve the instance you created in Problem 1.4 given the demand scenarios and probabilities listed in Table 6. All other instance data is listed in Tables 1— 4.

1.6 Problem
Solve the instance you created in Problem 1.5.

1.7 Problem
What is the Expected Value of Perfect Information (EVPI) for the solution to Problem 1.6?

1.8 Problem
What is the Value of the Stochastic Solution (VSS) VSS for the solution to Problem 1.6?

[2] If you miss this, *I'm* in trouble

s	p	t	d_{pts}
1	Tomato-Paste	1	100
1	Ketchup	1	30
1	Salsa	1	5
1	Tomato-Paste	2	100
1	Ketchup	2	30
1	Salsa	2	5
1	Tomato-Paste	3	100
1	Ketchup	3	30
1	Salsa	3	5
2	Tomato-Paste	1	100
2	Ketchup	1	30
2	Salsa	1	5
2	Tomato-Paste	2	100
2	Ketchup	2	30
2	Salsa	2	5
2	Tomato-Paste	3	200
2	Ketchup	3	40
2	Salsa	3	20
3	Tomato-Paste	1	100
3	Ketchup	1	30
3	Salsa	1	5
3	Tomato-Paste	2	200
3	Ketchup	2	40
3	Salsa	2	20
3	Tomato-Paste	3	100
3	Ketchup	3	30
3	Salsa	3	5
4	Tomato-Paste	1	100
4	Ketchup	1	30
4	Salsa	1	5
4	Tomato-Paste	2	200
4	Ketchup	2	40
4	Salsa	2	20
4	Tomato-Paste	3	200
4	Ketchup	3	40
4	Salsa	3	20

s	ρ_s
1	0.15
2	0.4
3	0.15
4	0.3

Table 6: Demand Scenarios and Probabilities for TTT Problem 1.5

Probabilistic Constraints

TTT management has now decided that it will no longer explicitly consider the fact that it can purchase extra resources. Instead, they will consider the amount of resource $r \in R$ available (b_r) as a random variable $b_r(\omega)$. TTT wants to enforce that the probability of exceeding the capacity $b_r(\omega)$ is sufficiently small for each resource and planning period.

1.9 Problem

Let $\theta_r, r \in R$ be the probability that TTT's production plan exceeds the capacity of resource r in a time period. Modify the model you created as an answer to Problem 1.4 to enforce TTT management's new requirements.

1.10 Problem

Solve the instance you created in Problem 1.9 as a chance-constrained problem, with values of θ_r and probability distributions for the demands b_r given in Table 7.[3]

r	$b_r(\omega)$	θ_r
Labor	$\mathcal{N}(200, 400)$	0.15
Tomatoes	$\mathcal{N}(250, 500)$	0.10
Sugar	$\mathcal{N}(300, 500)$	0.10
Spices	$\mathcal{N}(100, 100)$	0.02

Table 7: Distribution of Resource Capacity ($b_r(\omega)$) and Maximum Probability of Exceeding Capacity (θ_r).

[3] $\mathcal{N}(\mu, \sigma^2)$ is the Normal Distribution with mean μ and variance σ^2

2 Shapes

For each of the problems in this section you are to state the *shape* of the function—convex, concave, or neither convex nor concave—and prove your statement.

Let $x \in \Re^n, \eta \in \Re^p$. Consider the function

$$Q(x, \eta) = \min_{y \in \Re^p_+} \{\eta^T y | Wy = h - Tx\}.$$

2.1 Problem

For fixed $\hat{x}$, what is the shape of $Q(\hat{x}, \eta)$?

2.2 Problem

For fixed $\hat{\eta}$, what is the shape of $Q(x, \hat{\eta})$?

2.3 Problem

What is the shape of the function $Q(x, \eta)$?

3 The Last Stochastic Programming Problem Ever!

In two-stage stochastic LP with recourse, consider the following second stage problem:

$$Q(x, \omega) = \min_y \{y \mid y \geq \omega, y \geq x\}.$$

Assume that $x \geq 0$. Let ω have the probability density:

$$f(\omega) = \frac{2}{\omega^3}, \quad \omega \geq 1.$$

The Distribution Problem

3.1 Problem

Determine a closed form expression for $\mathcal{Q}(x) = \mathbb{E}_\omega Q(x, \omega)$.

3.2 Problem

Solve the problem

$$z^* = \min_{x \geq 0} \{x + \mathcal{Q}(x)\}.$$

SAA

Simulation Knowledge:

A general approach to generating a random variable ω drawn from a continuous probability distribution F is the following:

1. Generate $u \approx \mathcal{U}(0, 1)$
2. Return $\omega = F^{-1}(u)$

3.3 Problem

Create a sample average approximation (SAA) to the function $f(x) = x + \mathcal{Q}(x)$ using $N = 7$ samples.

3.4 Problem

Let $\hat{f}(x)$ be the SAA you created in Problem 3.3. Solve the problem

$$T^* = \min_{x \geq 0} \{\hat{f}(x)\}.$$

Let $\chi = \arg\min_{x \geq 0} \{\hat{f}(x)\}$

3.5 Problem

How (in general) is T^* related to z^* (the optimal objective function value in Problem 3.2).

3.6 Problem

What is $f(\chi)$? How does $f(\chi)$ relate to z^*?

3.7 Problem

Bonus:[4] Suppose you wish to use sampling to obtain an estimate of an upper bound with an associated confidence interval for $f(\chi)$. Describe how you might do this or explain why you can't.

[4] This is a Trick Question