IE 495 – Stochastic Programming
Problem Set #1 — Solutions

1 Random Linear Programs and the Distribution Problem

Recall the random linear program that we saw in class:

minimize

$$x_1 + x_2$$

subject to

$$\begin{aligned} \omega_1 x_1 + x_2 &\geq 7 \\ \omega_2 x_1 + x_2 &\geq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

with $\omega_1 \sim \mathcal{U}[1,4]$ and $\omega_2 \sim \mathcal{U}[1/3,1]$.
Let

- $(x_1^*(\omega), x_2^*(\omega))$ be the optimal solution for a given value of $\omega = (\omega_1, \omega_2)$.
- $v^*(\omega) = x_1^*(\omega) + x_2^*(\omega)$ be the optimal objective function value.

1.1 Problem

Calculate $x_1^*(\omega), x_2^*(\omega)$, and $v^*(\omega)$ for all $\omega \in \Omega = [1,4] \times [1/3,1]$.

Answer:
There are six basic solutions. $(0,0)$ and $(0,4)$ are not basic feasible solutions. By inspection, we can exclude the points $(0,7)$ and $(4/\omega_1, 0)$ as optimal solutions. There are therefore two cases:

- The optimal solution occurs at the intersection of $x_2 = 0$ and $\omega_1 x_1 + x_2 = 7$. This occurs when $7/\omega_1 >= 4/\omega_2$.
- The optimal solution occurs at the intersection of the inequalities $\omega_1 x_1 + x_2 = 7$ and $\omega_2 x_1 + x_2 = 4$.

Putting these two facts together, we can conclude that

$$x_1^* = \begin{cases} \frac{3}{\omega_1 - \omega_2} & \text{if} \frac{7}{\omega_1} \leq \frac{4}{\omega_2} \\ \frac{7}{\omega_1} & \text{Otherwise} \end{cases}$$

$$x_2^* = \begin{cases} \frac{4\omega_1 - 7\omega_2}{\omega_1 - \omega_2} & \text{if} \frac{7}{\omega_1} \leq \frac{4}{\omega_2} \\ 0 & \text{Otherwise} \end{cases}$$

$$v^* = \begin{cases} \frac{4\omega_1 - 7\omega_2 + 3}{\omega_1 - \omega_2} & \text{if} \frac{7}{\omega_1} \leq \frac{4}{\omega_2} \\ \frac{7}{\omega_1} & \text{Otherwise} \end{cases}$$

Q.E.D.

1.2 Problem

Calculate the distribution function F_{v^*} of the random variable $v^*(\omega)$.

Answer:

This one was *hard*. In general, determining the distribution function of a sum of random variables is an exercise in messy integration, which is all this was.

Let

$$A_{I_x} = \{(\omega_1, \omega_2) | 7/\omega_1 \leq 4/\omega_2, 4\omega_1 - 7\omega_2 + 3 \leq x(\omega_1 - \omega_2)$$

Let

$$A_{II_x} = \{(\omega_1, \omega_2) | 7/\omega_1 \geq 4/\omega_2, 7/\omega_1 \leq x$$

We must compute

$$F_{v^*}(x) = P(v^* \leq x) = \int_{A_{I_x}} f(\omega_1) f(\omega_2) d\omega_1 d\omega_2 + \int_{A_{II_x}} f(\omega_1) f(\omega_2) d\omega_1 d\omega_2$$

Note that $f(\omega_1) = 1/3, f(\omega_2) = 3/2$, so $f(\omega_1)f(\omega_2) = 1/2$. Therefore, the integral is just a matter of simple algebra and geometry. Consider the picture in Figure 1. In ω-space, as x increases, the line swings from top to bottom.

First we'll worry about A_{I_x}. Consider the picture drawn in Figure 2. The points specified by A, B, C are

Point	ω_1 coord.	ω_2 coord
A	$7/x$	$4/x$
B	$(2+x)/3(x-4)$	$1/3$
C	$7/4$	1

The picture is only valid when the intersection of the lines $(x-4)\omega_1 + (7-x)\omega_2 = 3$ and $\omega_2 = 1/3$ is ≤ 4.

$$\omega_1 = \frac{2+x}{3(x-4)} \leq 4 \Leftrightarrow x \geq 50/11$$

Figure 1: Integration Area

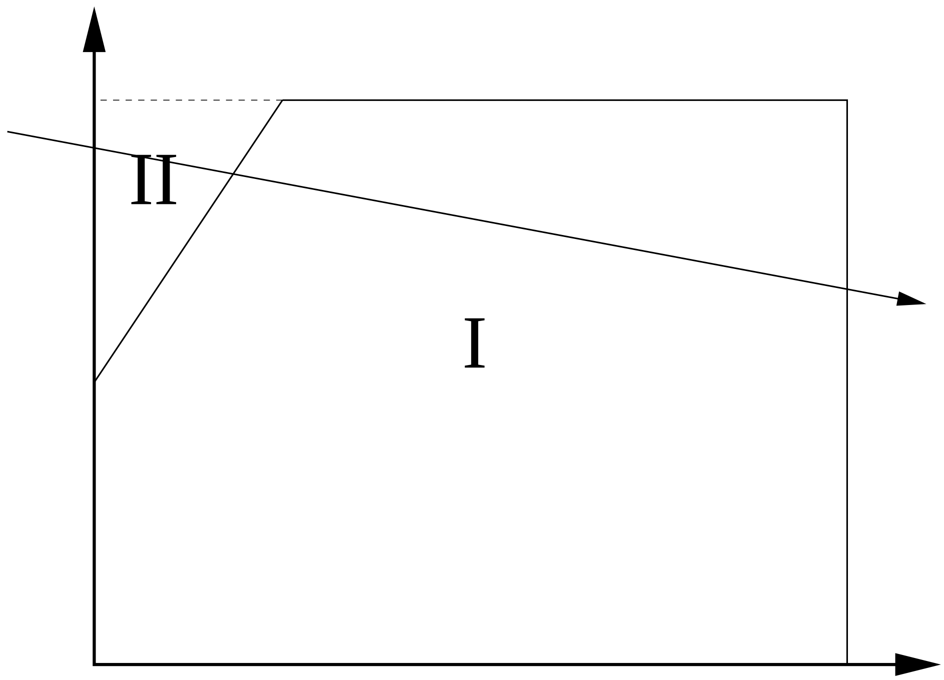
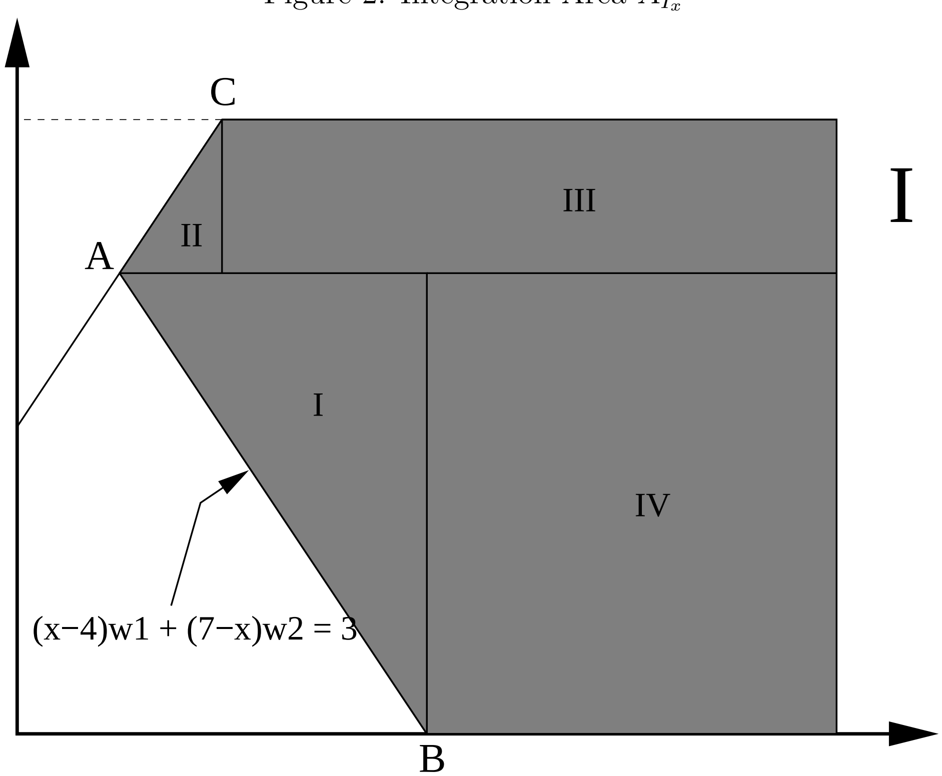


Figure 2: Integration Area A_{I_x}



So the following formula we will derive for A_{I_x} is valid only when $x \geq 50/11$.

We simply need to compute the area. I have done it in the following table.

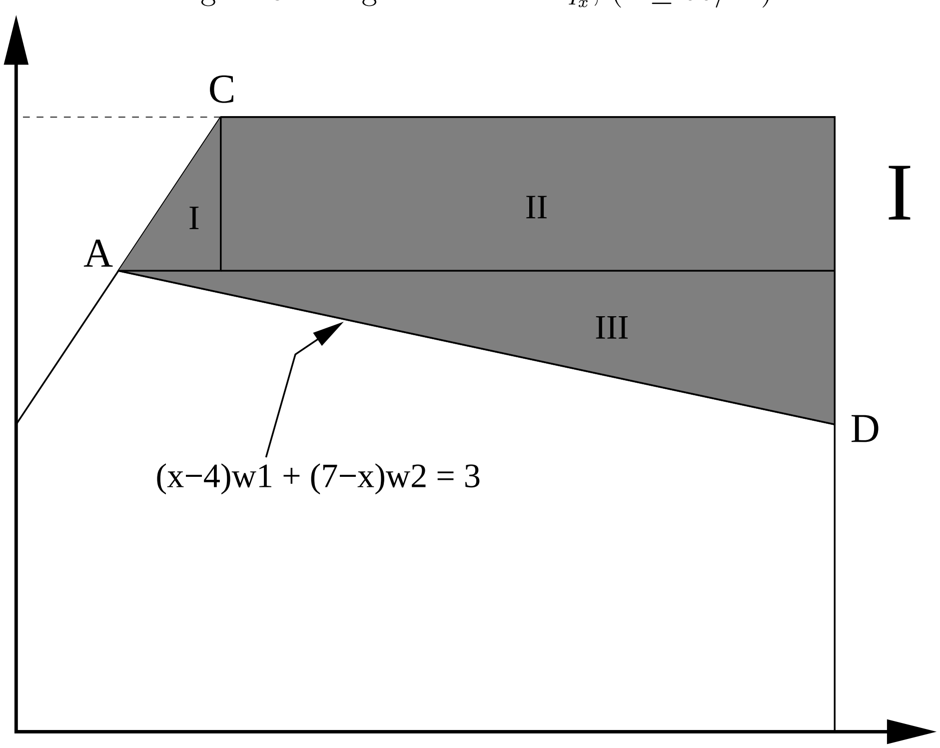
$$\begin{array}{rc} I: & 1/2\left(\frac{2+x}{3(x-4)} - \frac{7}{x}\right)\left(\frac{4}{x} - \frac{1}{3}\right) \\ II: & 1/2\left(\frac{7}{4} - \frac{7}{x}\right)\left(1 - \frac{4}{x}\right) \\ III: & \left(4 - \frac{7}{4}\right)\left(1 - \frac{4}{x}\right) \\ IV: & \left(4 - \frac{2+x}{3(x-4)}\right)\left(\frac{4}{x} - \frac{1}{3}\right) \end{array}$$

Adding these up[1] gives

$$A_{I_x} = \frac{1}{72}\left(\frac{-432 - 472x + 133x^2}{x(x-4)}\right) \qquad x \geq 50/11$$

If $x \leq 50/11$ the area we must integrate resembles that shown in Figure 3.

Figure 3: Integration Area A_{I_x}, $(x \leq 50/11)$



Point	ω_1 coord.	ω_2 coord
A	$7/x$	$4/x$
C	$7/4$	1
D	4	(4x-19)/(x-7)

[1] Thank Heavens for Maple!

$$\begin{array}{rc} I: & 1/2\left(\frac{7}{4}-\frac{7}{x}\right)\left(1-\frac{4}{x}\right) \\ II: & \left(4-\frac{7}{4}-\right)\left(1-\frac{4}{x}\right) \\ III: & 1/2\left(4-\frac{7}{x}\right)\left(\frac{4}{x}-\frac{4x-19}{x-7}\right) \end{array}$$

Adding these up yields

$$A_{I_x} = \frac{3}{8}\left(\frac{13x^2-59x+28}{x(7-x)}\right) \qquad x \leq 50/11$$

To finish, we must compute

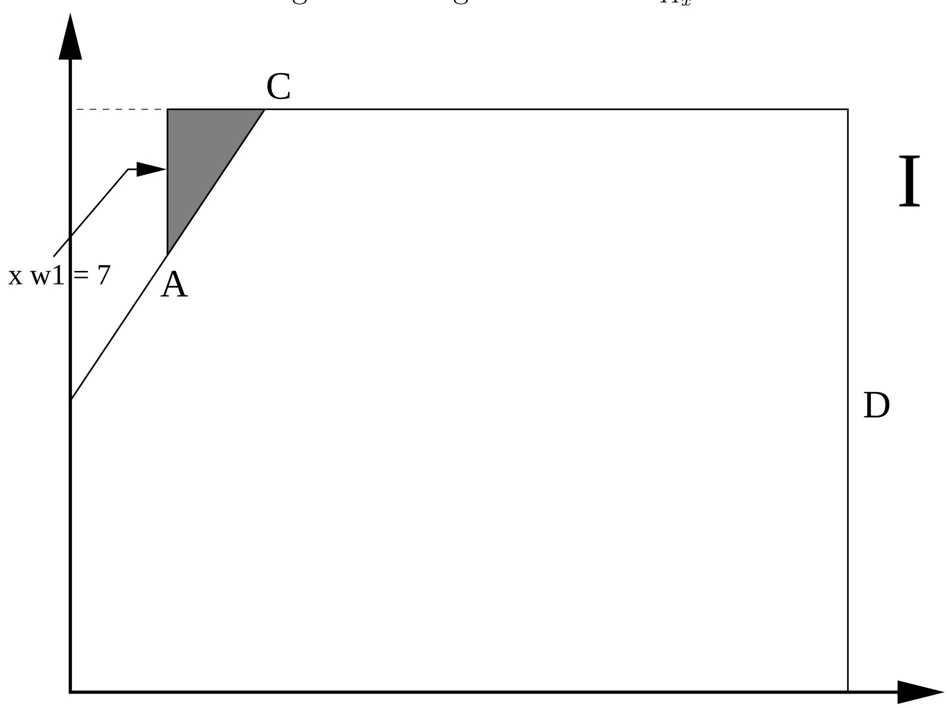
$$\int_{A_{II_x}} f(\omega_1)f(\omega_2)d\omega_1 d\omega_2,$$

where

$$A_{II_x} = \{(\omega_1,\omega_2)|7/\omega_1 \geq 4/\omega_2, 7/\omega_1 \leq x\}.$$

This integration area is shown in Figure 4.

Figure 4: Integration Area A_{II_x}



$$A_{II_x} = \frac{1}{2}\left(\frac{7}{4}-\frac{7}{x}\right)\left(1-\frac{4}{x}\right) = \frac{7(x-4)^2}{8x^2}$$

Putting everything together we have

$$F_{v^*}(x) = 0 \qquad 0 \leq x \leq 4.$$

If $4 \leq x \leq 50/11$,

$$\begin{aligned} F_{v^*}(x) &= \frac{1}{2}(A_{I_x} + A_{II_x}) = \frac{3}{8}\left(\frac{13x^2 - 59x + 28}{x(7-x)}\right) + \frac{7(x-4)^2}{8x^2} \\ &= \frac{8x^3 - 18x^2 - 105x + 196}{4x^2(7-x)} \end{aligned}$$

If $50/11 \leq x \leq 7$, we have

$$\begin{aligned} F_{v^*}(x) &= \frac{1}{2}(A_{I_x} + A_{II_x}) = \frac{1}{72}\left(\frac{-432 - 472x + 133x^2}{x(x-4)}\right) + \frac{7(x-4)^2}{8x^2} \\ &= \frac{4x^3 + 201x^2 - 1260x + 1568}{16x^2(7-x)} \end{aligned}$$

$$F_{v^*}(x) = 1 \qquad x \geq 7$$

Q.E.D.

1.3 Problem

Calculate the density of $v^*(\omega)$.

Answer:
Using the relation

$$f_{v^*}(x) = \frac{d}{dx} F_{v^*}(x)$$

and Maple to compute the derivatives for us, we find that for $4 \leq x \leq 50/11$ we have

$$f_{v^*}(x) = \frac{38x^3 - 210x^2 + 1323x - 2744}{4x^3(x-7)^2}$$

For $50/11 \leq x \leq 7$, we have

$$f_{v^*}(x) = \frac{37x^3 - 432x^2 + 1872x - 2688}{12x^3(x-4)^2}$$

$$f_{v^*}(x) = 0 \qquad \text{elsewhere}$$

Q.E.D.

1.4 Problem

Calculate $\mathbb{E}_\omega[v^*(\omega)]$.

Answer:

$$\begin{aligned}\mathbb{E}_\omega[v^*(\omega)] &= \int_{x=-\infty}^{\infty} x f_{v^*}(x)dx = \int_{x=4}^{7} x f_{v^*}(x) \\ &= \int_{x=4}^{50/11} x \frac{38x^3 - 210x^2 + 1323x - 2744}{4x^3(x-7)^2} dx \\ &\quad + \int_{x=50/11}^{7} x \frac{37x^3 - 432x^2 + 1872x - 2688}{12x^3(x-4)^2} dx\end{aligned}$$

Once again, we thank heaven for Maple, and we compute the integral as

$$\mathbb{E}_\omega[v^*(\omega)] = 4.752665590$$

This answer was confirmed with a Monte Carlo experiment, sampling 100,000 different points (ω_1, ω_2). The integral was estimated to be

$$\widehat{\mathbb{E}_\omega}[v^*(\omega)] = 4.75238$$

with a standard error of ± 0.002. This took *a lot* less time than explicitly calculating the integral. Q.E.D.

2 Proofs

To answer these questions, you will need to know the following definitions.

1 Definition
A function $f : D(f) \mapsto R(f)$ is *Lipschitz continuous* on its domain $D(f)$ if and only if there exists a constant L (called the Lipschitz constant) such that

$$|f(a) - f(b)| \leq L|a - b|$$

for all $a, b \in D(f)$

Note: The definition of Lipschitz continuity on page 78 of Birge and Louveaux is just plain wrong.

2 Definition
A set $S \subseteq \Re^n$ is polyhedral if and only if it can be written as

$$S = \{x \in \Re^n : Ax \leq b\}$$

for a matrix $A \in \Re^{m \times n}$ and vector $b \in \Re^m$.

For each of the problems, state whether they are true or false. In either case, provide a proof. (If the statement is false, then a counterexample is such a proof).

2.1 Problem

The intersection of convex sets is convex.

Answer:
This statement is **true**. It suffices to show that it is true for two convex sets A and B. Let $x \in A \cap B$ and $y \in A \cap B$. Consider the point $z \equiv \lambda x + (1-\lambda)y$ for any $\lambda \in [0,1]$. Since A is convex, and $x, y \in A$, $z \in A$. Since B is convex and $x, y \in B$, $z \in B$, so $z \in A \cap B$, which completes the proof. Q.E.D.

2.2 Problem

The union of convex sets is convex

Answer:
This statement is **false**. The line segments $A = [0,1], B = [2,3]$ are convex sets in $\Re$. For the points $1 \in A \cup B$ and $2 \in A \cup B$, the point $0.5(1) + 0.5(2) = 3/2 \notin A \cup B$. Q.E.D.

2.3 Problem

All convex functions are Lipschitz continuous on their domain.

This statement is **false.** Consider the following function defined on $\Re_+$.

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

This function is convex. (It's epigraph is certainly a convex set). However, for $a = 0$,

$$\lim_{b \to 0^+} \frac{|f(a) - f(b)|}{a - b} \to \infty,$$

so there does not exist a finite Lipschitz constant L.

Many of you provided nice proofs of the above statement if "on their domain" is replaced with "on the interior of their domain". (Or if D is open).

2.4 Problem

All polyhedral sets are convex.

Answer:
This statement is **true.** Let $x_1, x_2 \in S$, so that $Ax_1 \leq b, Ax_2 \leq b$. Consider the point $z = \lambda x_1 + (1-\lambda)x_2$ for some $\lambda \in [0,1]$.

$$Az = A(\lambda x_1 + (1-\lambda)x_2) = \lambda A x_1 + (1-\lambda)Ax_2 \leq \lambda b + (1-\lambda)b = b,$$

so $Az \in S$, and the polyhedron S is convex.

Q.E.D.

2.5 Problem

Let $f : \Re^n \mapsto \Re$ be a convex function. Then for any $x_1, x_2, \ldots, x_k$, and all sets of "convex multipliers" $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0 \ \forall i = 1, 2, \ldots, k$,

$$f(\sum_{i=1}^k \lambda_i x_i) \leq \sum_{i=1}^k \lambda_i f(x_i).$$

Answer:

The statement is true for $k = 2$, since it precisely the definition of a convex function. The proof is by induction. Assume the statement true for k. We will show that this implies that it is also true for $k + 1$.

If $\lambda_{k+1} = 1$, then the statement is trivially true for $k + 1$. So assume that $\lambda_{k+1} < 1$. For $i = 1, 2, \ldots, k$ define

$$\bar{\lambda}_i = \frac{\lambda_i}{1 - \lambda_{k+1}}.$$

Note first that

$$\sum_{i=1}^k \bar{\lambda}_i = 1,$$

since

$$\frac{\sum_{i=1}^k \lambda_i + \lambda_{k+1}}{1 - \lambda_{k+1}} = \frac{1}{1 - \lambda_{k+1}} \Rightarrow \sum_{i=1}^k \bar{\lambda}_i = 1.$$

It is not necessary for this proof, but note that the point $z = \sum_{i=1}^k \bar{\lambda}_i x_i$ lies in the convex hull of the points x_i, so that this inequality would hold under even stonger conditions on the domain of f.

$$\sum_{i=1}^{k+1} \lambda_i x_i = (1 - \lambda_{k+1}) \left[\sum_{i=1}^k \frac{\lambda_i x_i}{1 - \lambda_{k+1}} \right] + \lambda_{k+1} x_{k+1} = (1 - \lambda_{k+1}) z + \lambda_{k+1} x_{k+1}.$$

Therefore,

$$
\begin{aligned}
f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) &= && (1)\\
f((1-\lambda_{k+1})z + \lambda_{k+1}x_{k+1}) &\leq (1-\lambda_{k+1})f(z) + \lambda_{k+1}f(x_{k+1}) \\
&= (1-\lambda_{k+1})f\left(\sum_{i=1}^{k} \frac{\lambda_i x_i}{1-\lambda_{k+1}}\right) + \lambda_{k+1}f(x_{k+1}) \\
&\leq (1-\lambda_{k+1})\left(\sum_{i=1}^{k} \frac{\lambda_i}{1-\lambda_{k+1}} f(x_i)\right) + \lambda_{k+1}f(x_{k+1}) && (2)\\
&= \sum_{i=1}^{k+1} \lambda_i f(x_i)
\end{aligned}
$$

Step 2 is using the definition of a convex function, and step 2 uses the induction hypothesis. This completed the proof. Q.E.D.

3 Farmer Ted

3.1 Problem

Birge and Louveaux #1.3, Page 18.

Answer:

Of course there are many valid formulations, and I don't claim that this one is best. Make the following definitions:

- C : Set of crops
- S : Set of scenarios
- F : Set of fields
- x_c : Acres of crop $c \in C$ to plant
- w_{cs} : Tons of crop $c \in C$ to sell at favorable price in scenario $s \in S$
- e_{cs} : Tons of crop $c \in C$ to sell at unfavorable price in scenario $s \in S$
- y_{cs} : Tons of crop $c \in C$ to purchase to meet quota in scenario $s \in S$.
- z_{cf} : Binary indicator variable on whether we plant crop $c \in C$ in field $f \in F$.
- α_{cs} : Yield of crop $c \in C$ in scenario $s \in S$.
- γ_c : Planting cost of crop $c \in C$
- H_c : Favorable sales price of crop $c \in C$
- L_c : Unfavorable sales price of crop $c \in C$
- Q_c : Quota for crop $c \in C$ (Max amount thta can be sold at favorable price)
- η_c : Purchase price of crop $c \in C$
- R_c : Min requirement for crop $c \in C$
- A_f : Area of field $f \in F$
- K : Total number of acres
- p_s : Probability of yield scenario $s \in S$.

maximize

$$-\sum_{c \in C} \gamma_c x_c + \sum_{sinS} p_s \sum_{c \in C} (H_c w_{cs} + L_c + e_{cs} - \eta_c y_{cs}$$

subject to

$$\begin{aligned}
\sum_{c \in C} x_c &\leq K \\
x_c &= \sum_{f \in F} A_f z_{cf} \qquad \forall c \in C \\
\sum_{c \in C} z_{cf} &\leq 1 \qquad \forall f \in F \\
\alpha_{cs} x_c + y_{cs} - w_{cs} - e_{cs} &= R_c \\
w_{cs} &\leq Q_c \qquad \forall c \in C, \forall s \in S
\end{aligned}$$

3.2 Problem

Birge and Louveaux #1.5, Page 18.

Answer:

Stochastic Programming Solution

My AMPL files are on the course web page.

```
http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts.mod
http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts.dat
```

The optimal solution is

```
ampl: solve;
CPLEX 7.1.0: optimal integer solution; objective 107974.7811
34 MIP simplex iterations
0 branch-and-bound nodes

ampl: display x;
x [*] :=
Beans  250
 Corn  105
Wheat  145
;
```

The non-zero values of z are

```
ampl: display z;
z :=
Beans 1   1
Beans 4   1
Corn  3   1
Wheat 2   1
;
```

The optimal policy is to plant beans in fields 1 and 4, corn in field 3, and wheat in field 2.

Expected Value of Perfect Information

If we knew *a priori* what the yields would be, we could make the following amounts of money:

Scenario	z^*
Good Yields	167390
Average Yields	114875
Bad Yields	57640

Each of the yield scenarios are equally likely, so we have

$$\text{EVPI} = \frac{1}{3}(167390 + 114875 + 57640) - 107975 = 5326.67$$

Computational Note:

To solve the various scenario models in AMPL, I simply modified the data file to have only one scenario. For example, for the "good yield" scenario, my data file was the following:

```
set C := Wheat Corn Beans;
set F := 1 2 3 4;

set S := 1;

param p :=
1 1.0;

param Yield :=
Wheat 1 3
Corn 1 3.6
Beans 1 24 ;
```

```
param:    PlantingCost SalePrice :=
   Wheat       150         170
   Corn        230         150
   Beans       260          36 ;

param: MaxAtGoodPrice LowSalePrice :=
   Beans 6000 10;

param: PurchasePrice MinReq :=
   Wheat 238 200
   Corn  210 240 ;

param TotalAcres := 500;

param Area :=
1 185
2 145
3 105
4 65 ;
```

Q.E.D.

Value of Stochastic Solution

Answer:
The planting policy based on an average yield scenario is to plant the following:

Crop	Acres
Beans (Beats)	290
Corn	65
Wheat	145

Now we must simply compute the optimal recourse in each of the scenarios given that this policy is fixed. An easy way to do this is to change the x (planting) variables in the model file to be a parameter, and in the data file we fix the

```
# Number of acres of crop c to plant
param x{C} >= 0;

param x :=
Wheat 145
Corn 65
Beans 290 ;
```

You can find my AMPL model and data files are

http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts-vss.mod
http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts-vss.dat

Making these changes yields and resolving the model yields:

```
ampl: model fts-vss.mod;
ampl: data fts-vss.dat;
ampl: solve;
CPLEX 7.1.0: optimal integer solution; objective 106554.7813
4 MIP simplex iterations
0 branch-and-bound nodes
```

So then

$$\text{VSS} = 107975 - 106555 = 1420.$$

Q.E.D.

3.3 Problem

Birge and Louveaux #1.7, Page 19.

Answer:

Note, for these problems, we do not have the one-crop-per-field constraints present in the previous model. (Or at least thats how I read the problem).

Worst Case Planning Profit Loss

Answer:

The worst-case solution (taken from Table 4 on Page 7) is the following:

Crop	Acres
Wheat	100
Corn	25
Beans	375

The objective value when x is fixed to the solution above is 86599.79515
So the profit loss is

$$\text{Profit Loss} = 108390 - 86600 = 21790$$

I used

http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts-vss-lp.mod
http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts-vss-lp.dat

to compute this answer.

Worst Case Profit $>= 58000$

We simply add the constraint that the profit is $>= 58000$ $\forall s \in S$.

I used the files

```
http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts-lp.mod
http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts-lp.dat
```

And the expected profit lost is $7214.03.

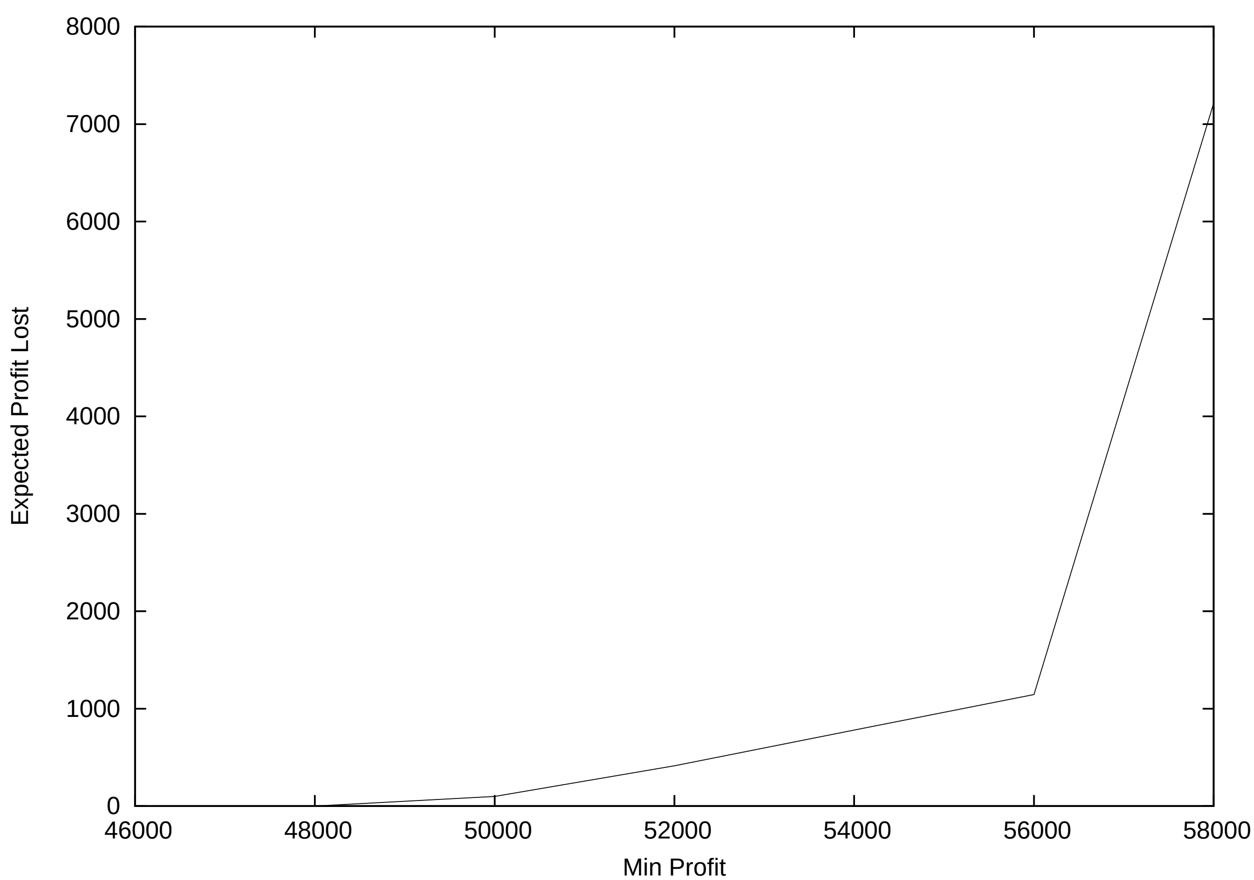
Worst Case Profit Graph

Answer:

I used the commands file

```
http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/fts.run
```

to make the following picture:



AMPL output is shown below:

```
ampl: commands fts.run;


CPLEX 7.1.0: optimal solution; objective 101175.9724
14 simplex iterations (12 in phase I)
Profit = 101176

108390 - Profit = 7214.03

x [*] :=
Beans  328.571
 Corn   71.4286
Wheat  100
;

CPLEX 7.1.0: optimal solution; objective 107245.8092
4 simplex iterations (1 in phase I)
Profit = 107246

108390 - Profit = 1144.19

x [*] :=
Beans  293.651
 Corn  100
Wheat  106.349
;

CPLEX 7.1.0: optimal solution; objective 107610.8899
0 simplex iterations (0 in phase I)
Profit = 107611

108390 - Profit = 779.11

x [*] :=
Beans  277.778
 Corn  100
Wheat  122.222
;

CPLEX 7.1.0: optimal solution; objective 107975.9707
0 simplex iterations (0 in phase I)
```

```
Profit = 107976

108390 - Profit = 414.029

x [*] :=
Beans  261.905
 Corn  100
Wheat  138.095
;

CPLEX 7.1.0: optimal solution; objective 108291.4484
2 simplex iterations (1 in phase I)
Profit = 108291

108390 - Profit = 98.5516

x [*] :=
Beans  250
 Corn   94.0476
Wheat  155.952
;

CPLEX 7.1.0: optimal solution; objective 108389.7827
1 simplex iterations (1 in phase I)
Profit = 108390

108390 - Profit = 0.21729

x [*] :=
Beans  250
 Corn   80
Wheat  170
;
```

Q.E.D.

4 Yield Management

For the problem described by Birge and Louveaux, Page 43, Number 1...

4.1 Problem

Develop the stochastic programming model.

Answer:

Make the following definitions:

- C : Set of fare classes
- S : Set of demand scenarios
- x_c : Number of seats reserved for fare class $c \in C$
- y_{cs} : Number of seats of fare class $c \in C$ purchased in scenario $s \in S$
- α_c : Number of seats used by fare class $c \in C$.
- γ_c : (Relative) profit obtained by selling one seat of class $c \in C$
- D_{cs} : Demand for seats of class $c \in C$ under scenario $s \in S$.
- p_s : Probability of demand scenario $s \in S$ occuring.
- K : Number of seats on the plane

Then the following is a valid formulation that will maximize the expected profit for Northam Airlines. maximize

$$\sum_{sinS} \sum_{cinC} p_s \gamma_c y_{cs}$$

subject to

$$\begin{aligned}
\sum_{c \in C} \alpha_c x_c &\leq K \\
y_{cs} &\leq x_c \qquad \forall c \in C, \forall s \in S \\
y_{cs} &\leq d_{cs} \qquad \forall c \in C, \forall s \in S \\
x_c &\geq 0 \qquad \forall c \in C \\
y_{cs} &\geq 0 \qquad \forall c \in C, \forall s \in S
\end{aligned}$$

Q.E.D.

4.2 Problem

Solve the stochastic program.

Answer:

I used the model and data files found at..

`http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/yield.mod`
`http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/yield.dat`

The solution is the following:

```
ampl: model yield.mod;
ampl: data yield.dat;
ampl: solve;
CPLEX 7.1.0: optimal solution; objective 208.333125
15 simplex iterations (0 in phase I)
ampl: display x;
x [*] :=
Business   20
 Economy  150
   First   10
;
```

Q.E.D.

4.3 Problem

Calculate the value of the stochastic solution (VSS).

Answer:

For there problems I used `http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/yield-vss.mod` `http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/yield-vss.dat`

The mean value solution is

```
ampl: model yield-vss.mod;
ampl: data yield-vss.dat;
ampl: solve;
CPLEX 7.1.0: optimal integer solution; objective 225
4 MIP simplex iterations
0 branch-and-bound nodes
ampl: display x;
x [*] :=
Business   28
 Economy  136
   First   11
;
```

Not surprisingly, the policy is to allocate seats to meet average first class demand, then seats to meet average business demand, then put the rest in economy. Fixing this policy and evaluating the expected recourse function gives...

Fixing the solution to the value (11, 28, 136) and determining the average recourse cost gives...

```
ampl: solve;
CPLEX 7.1.0: optimal integer solution; objective 203.999796
0 MIP simplex iterations
0 branch-and-bound nodes
ampl: display x;
x [*] :=
Business   28
 Economy  136
   First   11
;
```

So

$$\text{VSS} = 208.3 - 204 = 4.333$$

Q.E.D.

5 Hospital Staffing

For the problem described by Birge and Louveaux, Page 46, Number 6[2]...

5.1 Problem

Develop a reasonable stochastic programming model.

Answer:

Define the following variables:

[2] Note – I believe there is an error in the book. When solving this problem, use (100, 90, *120*) as the first of the 9 demand scenarios

S	Set of scenarios
p_s	Probability of scenario $s \in S$
d_{as}	Number of patients that show up on Saturday in scenario $s \in S$
d_{us}	Number of patients that show up on Sunday in scenario $s \in S$
d_{ms}	Number of patients that show up on Monday in scenario $s \in S$
x_a	Number of nurses originally scheduled to work on Saturday
x_u	Number of nurses originally scheduled to work on Sunday
x_m	Number of nurses originally scheduled to work on Monday
y_s	Number of nurses called in to work the Sunday/Monday shift in scenarion $s \in S$
e_{as}	Extra number of patients on Saturday in scenario $s \in S$
e_{us}	Extra number of patients on Sunday in scenario $s \in S$
e_{ms}	Extra number of patients on Monday in scenario $s \in S$

Then a stochastic programming instance that will determine the number of nurses to staff on each day is given as follows:

minimize

$$300(x_a + x_u + x_m) + \sum_{s \in S} p_s(800y + 50(e_a + e_u + e_m))$$

subject to

$$\begin{array}{rcll} 10x_a + e_{as} & \geq & d_{as} & \\ 10x_u + y_s + e_{us} & \geq & d_{us} & \\ 10x_m + y_s + e_{ms} & \geq & d_{ms} & \\ y_{s_1} & = & y_{s_2} & \forall s_1, s_2 \in S \text{ such that } d_{as_1} = d_{as_2} \\ x_a, m_u, x_m, y, e_a, e_u, e_m & \geq & 0 & \end{array}$$

Note: I assumed that the number of nurse we decide to bring in *must* be a function of the demand we observed on Saturday. If you do not have this constraint, it is like you are able to observe the demand on Sunday, and then bring nurses in to help service the demand.

Note2: No one was very close on this one — I think becasue the problem is worded in a vague manner. I graded leniently... Q.E.D.

5.2 Problem

Solve the stochastic program.

Answer:

AMPL model and data files can be found at

`http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/nurse.mod`
`http://www.lehigh.edu/ jtl3/teaching/ie495/hw1/nurse.dat`

I found the optimal solution to be

Day	Nurses
Sat	9
Sun	9
Mon	10

With an expected cost of $9122.22

Q.E.D.

5.3 Problem

Calculate the value of the stochastic solution (VSS).

Answer:

The mean value solution (obtained by using `nurse.mod` with `nurse-avg.dat` is

Day	Nurses
Sat	9
Sun	9
Mon	11

Using the model `nurse-vss.mod` and the data file `nurse.dat` gives that the expected optimal recourse function for this solution has value 9200, so

$$VSS = 9200 - 9122 = \$78$$

Q.E.D.