# IE 495 – Stochastic Programming
# Problem Sets #5—#7

Due Date: April 28, 2003

Do the following problems. If you work alone, you will receive a 10% bonus on your score. These are the *final* homework sets for the semester. Problems 1–2 will be Problem Set #5, Problems 3–4 will be Problem Set #6, and Problems 5–6 will be Problem Set #7.

You are allowed to examine outside sources, but you must cite any references that you use. Please don't discuss the problems with other members of the class (other than your partner, if you are working with one).

## 1 Feasibility Cuts

The second stage constraints of a two-stage problem look as follows:

$$\begin{aligned} \begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & -1 & 2 & 1 \end{bmatrix} y &= \begin{bmatrix} -6 \\ -4 \end{bmatrix} \omega + \begin{bmatrix} 5 & -1 & 0 \\ 0 & 2 & 4 \end{bmatrix} x \\ y &\geq 0. \end{aligned}$$

Here $\omega$ is a random variable with support $\Omega = [0, 1]$.

### 1.1 Problem

Write down the linear programs (both primal and dual formulation(s)) needed to check whether or not there is a feasible second stage solution for a given $x$.

### 1.2 Problem

Describe how these formulations allow you to obtain an inequality that cuts of $x$, if there is no feasible second stage solution $y$ for that $x$.

### 1.3 Problem

Let $\hat{x} = (1, 1, 1)^T$. Find the inequality explicitly for this first stage solution $\hat{x}$.

# 2 Bounds

Consider our favorite random linear program.
minimize

$$\mathcal{Q}(x_1, x_2) = x_1 + x_2 + 5 \int_{\omega_1=1}^{4} \int_{\omega_2=1/3}^{1} y_1(\omega_1, \omega_2) + y_2(\omega_1, \omega_2) d\omega_1 d\omega_2$$

subject to

$$\begin{array}{rcll} \omega_1 x_1 + x_2 + y_1(\omega_1, \omega_2) & \geq & 7 & \forall \omega_1, \omega_2 \in \Omega \\ \omega_2 x_1 + x_2 + y_2(\omega_1, \omega_2) & \geq & 4 & \forall \omega_1, \omega_2 \in \Omega \\ x_1 & \geq & 0 & \\ x_2 & \geq & 0 & \\ y_1(\omega_1, \omega_2) & \geq & 0 & \forall \omega_1, \omega_2 \in \Omega \\ y_2(\omega_1, \omega_2) & \geq 0 & & \forall \omega_1, \omega_2 \in \Omega \end{array}$$

- $\Omega = \{\omega_1 \times \omega_2\}$
- $\omega_1 \sim \mathcal{U}[1, 4]$
- $\omega_2 \sim \mathcal{U}[1/3, 1]$

### 2.1 Problem

Compute the Jensen Lower Bound for $\mathcal{Q}(1, 3)$ using the partition $\Omega = \mathcal{S}^1 = \{\Omega\}$.

### 2.2 Problem

Compute the Edmundson-Madansky Upper Bound for $\mathcal{Q}(1, 3)$ using the partition $\Omega = \mathcal{S}^1 = \{\Omega\}$.

## Refining Bounds

Problems 2.3 and 2.4 depend on the following partition $\mathcal{S}^2$ of $\Omega$. Let

- $\Omega^I = \{\omega_1 \times \omega_2 | 1 \leq \omega_1 \leq 5/2, 1/3 \leq \omega_2 \leq 2/3\}$,
- $\Omega^{II} = \{\omega_1 \times \omega_2 | 5/2 \leq \omega_1 \leq 4, 1/3 \leq \omega_2 \leq 2/3\}$,
- $\Omega^{III} = \{\omega_1 \times \omega_2 | 1 \leq \omega_1 \leq 5/2, 2/3 \leq \omega_2 \leq 1\}$,
- $\Omega^I = \{\omega_1 \times \omega_2 | 5/2 \leq \omega_1 \leq 4, 2/3 \leq \omega_2 \leq 1\}$.

$$\mathcal{S}^2 = \{\Omega^I, \Omega^{II}, \Omega^{III}, \Omega^{IV}\}$$

See Figure 2 for a pictorial representation of the partition $\mathcal{S}^2$.

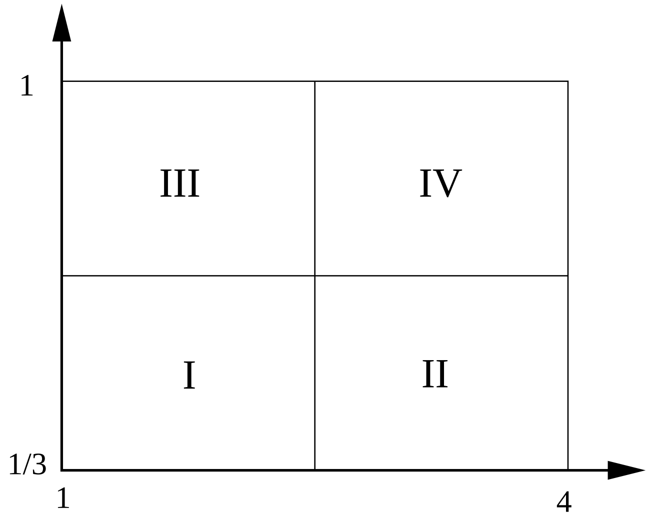
### 2.3 Problem

Compute the Jensen Lower Bound for $\mathcal{Q}(1, 3)$ using the partition $\mathcal{S}^2$

### 2.4 Problem

Compute the Edmundson-Madansky Upper Bound for $\mathcal{Q}(1, 3)$ using the partition $\mathcal{S}^2$.

Figure 1: Partition $\mathcal{S}^2$




## 3 Monte Carlo/Variance Reduction

Consider the following Stochastic LP.
$v^*$ = minimize

$$10x_1 + 7x_2 + 16x_3 + 6x_4 + \mathbb{E}_\omega \quad [\quad 40y_{11} + 45y_{21} + 32y_{31} + 55y_{41} + 24y_{12} + 27y_{22} + 19.2y_{32} + 33y_{42} + 4y_{13} + 4.5y_{23} + 3.2y_{33} + 5.5y_{43}]$$

subject to

$$\begin{aligned}
\sum_{i=1}^{4} y_{i1} &= \omega_1 \\
\sum_{i=1}^{4} y_{i2} &= \omega_2 \\
\sum_{i=1}^{4} y_{i3} &= \omega_3 \\
\sum_{i=1}^{4} x_i &\geq 12 \\
\sum_{j=1}^{3} y_{1j} &\leq x_1 \\
\sum_{j=1}^{3} y_{2j} &\leq x_2 \\
\sum_{j=1}^{3} y_{3j} &\leq x_3 \\
\sum_{j=1}^{3} y_{4j} &\leq x_4
\end{aligned}$$

$$\begin{aligned} 10x_1 + 7x_2 + 16x_3 + 6x_4 &\leq 120 \\ x, y &\geq 0 \end{aligned}$$

## Confidence Interval

Suppose the random variables $\omega_1, \omega_2, \omega_3$ are all independent with the following distributions.

- $\omega_1 \approx \mathcal{U}[3, 7]$
- $\omega_2 \approx \mathcal{U}[2, 3]$.
- $\omega_3 \approx \mathcal{U}[1, 2]$

### 3.1 Problem

Compute a (statistical) lower bound $L$ on $v^*$.[1]

### 3.2 Problem

Compute (and provide justification) for a 97.5% confidence interval around the value of $L$ you gave in Problem 3.1.[2]

### 3.3 Problem

Compute a (statistical) upper bound $U$ on $v^*$.[3]

### 3.4 Problem

Compute (and provide justification) for a 97.5% confidence interval around the value of $U$ you gave in Problem 3.3.[4]

[1] Bonus points given for a "tight" lower bound. (That is $v^* - L$ "small").

[2] Bonus Points given for producing the confidence interval with a "small" number of problems solved.

[3] Bonus points given for a "tight" upper bound. (That is $U - v^*$ "small").

[4] Bonus Points given for producing the confidence interval with a "small" number of problems solved.

# 4 Random Objective Coefficients and VSS

Consider the following two-stage stochastic LP with recourse:

$$\max_{x \in \Re^3_+} \{3x_1 + x_2 + 4x_3 + \mathbb{E}_\omega[q_1(\omega)y_1(\omega) + q_2(\omega)y_2(\omega)]\}$$

subject to

$$\begin{array}{rcll} x_1 + 2x_2 + y_1(\omega) - y_2(\omega) & \leq & 3 & \forall \omega \in \Omega \\ -x_1 - x_2 + x_3 + 2y_1(\omega) + 3y_2(\omega) & \leq & 1 & \forall \omega \in \Omega \\ x_2 + 3x_3 - y_1(\omega) + y_2(\omega) & \leq & 3 & \forall \omega \in \Omega \\ x_1, x_2, x_3 & \geq & 0 & \\ y_1(\omega), y_2(\omega) & \geq & 0 & \forall \omega \in \Omega \end{array}$$

In this example, $\omega$ is a random variable with finite support $\Omega$. Table 1 gives the realizations of $\omega$, the probabilities $p(\omega)$ of each realization, and the values of $q_1(\omega), q_2(\omega)$ in each realization.

| $\omega$ | $p(\omega)$ | $q_1(\omega)$ | $q_2(\omega)$ |
|---|---|---|---|
| 1 | 0.125 | -2 | -2 |
| 2 | 0.5 | 2 | 1 |
| 3 | 0.375 | 8 | 6 |

Table 1: Probability Distribution for Problem 4.

### 4.1 Problem
Compute the EVPI for this problem.

### 4.2 Problem
Compute the VSS for this problem.

## (Dis?)Proof Time

Let $X \subseteq \Re^n$ and $Y \subseteq \Re^p$ be polyhedral sets. Make the following definitions:

$$Q(x, \omega) = \min_{y(\omega) \in Y} \{q(\omega)^T y(\omega) | W y(\omega) = h - Tx\},$$

$$z_1^* = \min_{x \in X} \mathbb{E}_\omega[Q(x, \omega)],$$

$$z_2^* = \min_{x \in X} Q(x, \mathbb{E}_\omega[\omega]).$$

Assume that the matrices $T, W$ vectors $h, x$ and random vectors $y(\omega), q(\omega)$ all have appropriate dimension. Consider the following statement:

### 1 Statement

$$z_1^* = z_2^*.$$

### 4.3 Problem
What (in words) does Statement 1 mean?

**4.4 Problem**

Is Statement 1 true or false? If true, provide a proof. If false, provide a counter-example.

# 5 Stochastic IP – Recourse Function

Let

$$v(z) = \min_{y \in Y} \{2y_1 + 5y_2 + 6y_3 + y_4 | 2y_1 + 5y_2 + 7y_3 - y_4 = z\}, Y = \{\mathbb{Z}^3_+ \times \Re_+\}$$

$$\mathcal{Q}(x) = \mathbb{E}_\omega[v(h(\omega) - x)], x \in \Re$$

**5.1 Problem**

Let $h(\omega)$ have the following density.

- $P(h(\omega) = 5) = 1/3$
- $P(h(\omega) = 5.5) = 1/3$
- $P(h(\omega) = 6) = 1/3$

Draw the graph of $\mathcal{Q}(x)$ for $x \in [-10, 15]$.

**5.2 Problem**

Let $h(\omega)$ have the following density.

- $P(h(\omega) = 5) = 0.2$
- $P(h(\omega) = 5.25) = 0.2$
- $P(h(\omega) = 5.5) = 0.2$
- $P(h(\omega) = 5.75) = 0.2$
- $P(h(\omega) = 6) = 0.2$

Draw the graph of $\mathcal{Q}(x)$ for $x \in [-10, 15]$.

**5.3 Problem**

Let $h(\omega) \approx \mathcal{U}[5, 6]$.

Draw the graph of $\mathcal{Q}(x)$ for $x \in [-10, 15]$.

# 6 Stochastic IP – Integer LShaped Method

In this problem, we will be performing the Integer L-Shaped method on the following problem instance:

$$\min\{x + \mathcal{Q}(x), x \in \{0,1\}\},$$

where

$$\mathcal{Q}(x) = \mathbb{E}_\omega[v(x,\omega)],$$
$$v(x,\omega) = \min\{3/2y | y \geq \omega - x, y \in \mathbb{Z}_+\}$$

$$\begin{aligned} \mathrm{P}(\omega = 1.3) &= 0.5 \\ \mathrm{P}(\omega = 2.7) &= 0.5 \end{aligned}$$

An obvious lower bound for $\mathcal{Q}(x)$ is $L = 0$. Use this lower bound in all of the subsequent problems. Recall that an optimality cut at $x^k$ in the Integer L-Shaped Method is given as

$$\theta \geq (\mathcal{Q}(x^k) - L)\left(\sum_{j \in S^k} x_j - \sum_{j \notin S^k} x_j - |S^k| + 1\right) + L.$$

### 6.1 Problem

Draw the graph of $x + \mathcal{Q}(x)$ for all $x \in [0,1]$.

### 6.2 Problem

Starting with iteration ($k = 1$) in which $x^1 = 0$, compute $\mathcal{Q}(x^1)$ and the optimality cut associated with $x^1$. Draw the graph of the optimality cut on your graph from your answer to Problem 6.1.

### 6.3 Problem

Formulate and solve the master problem for iteration $k = 2$. Compute the optimal solution (call it $x^2$) to the master problem. Compute $\mathcal{Q}(x^2)$ and the optimality cut associated with $x^2$. Draw the graph of the optimality cut on your graph from your answer to Problem 6.2.

### 6.4 Problem

Formulate and solve the master problem for iteration $k = 3$. Compute the optimal solution (call it $x^3$) to the master problem. State what the next step of the algorithm will be.

### 6.5 Problem

Based on the discussion in your answer to Problem 6.4 state the optimal solution to the problem.