IE 495 – Lecture 13

Bounds in Stochastic Programming

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Outline

- Review
- Bounds
  - Distribution Problem
  - Numerical Integration
  - Lower Bound—Jensen's inequality
  - Tightening the Lower Bound
  - A numerical example

Review

- ⋆ Homework *not* due until Wed.
  - ◇ No homework assignment until after break.
  - ◇ You're welcome! (especially those of you taking your qualifying exam).

- Who wants to use high-performance computing?
- What is an "SMP" machine?
- What is Condor?
- Where is the fastest computer in the world located?

Bounds

- Think of the case in which we are trying to solve a stochastic program containing random variables that are drawn from a continuous distribution.

$$\min \mathcal{Q}(x) \equiv \mathbb{E}_\omega Q(x,\omega) = \int_\Omega Q(x,\omega) dF(\omega)$$

- Keep in mind that $\int_\Omega$ is one of those fancy Lebesgue-Stieltjes integrals, so it really is a multidimensional integral.
- For example, if $\Omega \subseteq \Re^3$,

$$\min \mathcal{Q}(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(x,\omega) dF(\omega)$$

Scary Looking!

$$\min_{x \in X} c^T x + \int \cdots \int_{\Omega} Q(x, \omega)$$

- Who knows how to solve optimization problems with integrals in them?
  - (NOT ME!)
- It is even very difficult to *evaluate* the function that you are trying to optimize.
- Things you can try...
- **1.** Solve the *distribution problem*.

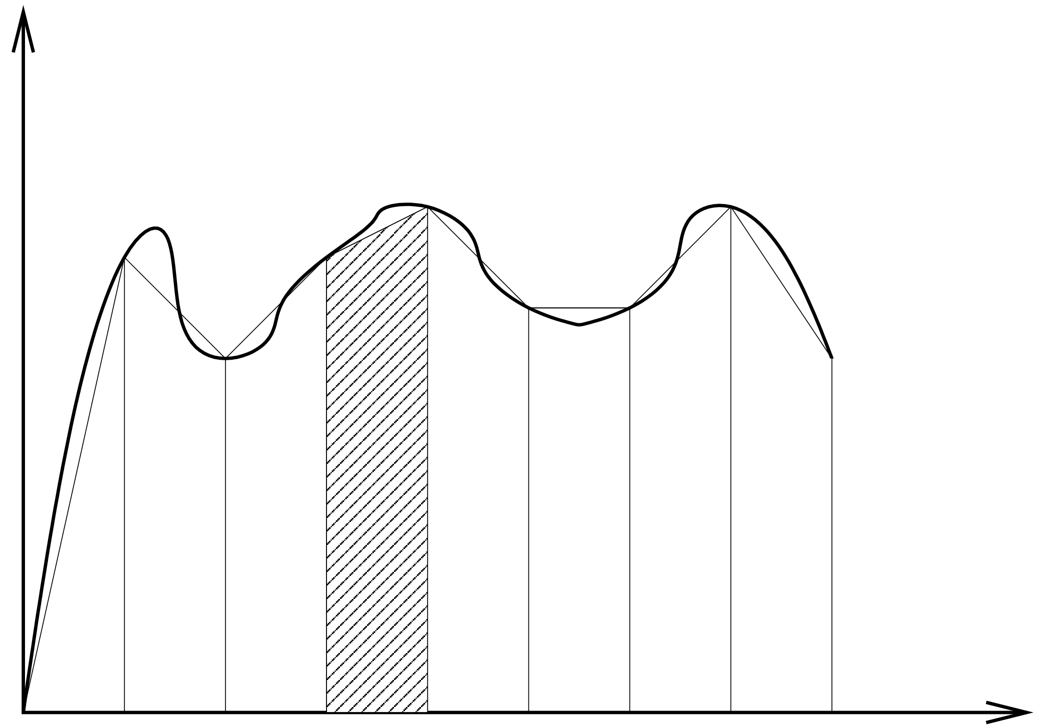
The Distribution Problem

- Develop a closed form expression for $Q(x, \omega)$
  - ◇ You obtain a solution to the recourse problem (for any value of x and realization ω) by inspection.
  - ◇ You have done this (or something similar) in HW#1, and HW#2.
  - ◇ Once you know a closed form for $Q(x, \omega)$, you just integrate away...

⋆ It is possible to obtain a closed form $Q(x, \omega)$ only for very simple problems.

Numerical Integration

- **2.** Another thing you can try is *numerical integration*.
- Trapezoid Rule? Simpson's rule?



Trapezoidal Rule

The n-point trapezoidal approximation to

$$\int_{x=a}^{b} f(x)dx \quad \text{with } \Delta x = \frac{b-a}{n}$$

is

$$T_n = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n))$$

Error Analysis

- People like to do numerical integration because it comes with fancy error analysis:

Thm:

If f'' is continuous on $[a, b]$ and $f'' \leq M \; \forall x \in [a, b]$, then

$$\max_{x \in [a,b]} \left| T_n - \int_a^b f(x)dx \right| \leq \frac{M(b-a)^3}{12}.$$

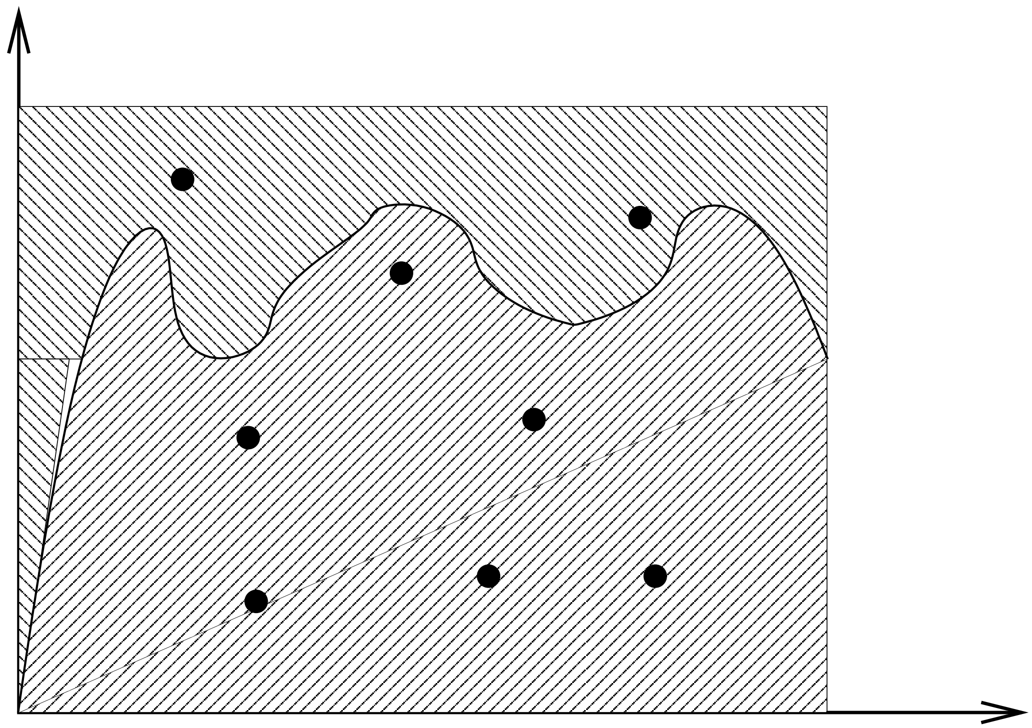
- What is wrong eith the above theorem in the case that $f(x) \equiv Q(x, \omega)$?

Numerical Integration

- $Q''(x, \omega)$ doesn't exist.
- If you want to know more about numerical integration, the buzzwords are...
  - ◇ Numerical quadrature, Simpson's rule...
- Numerical integration really only works well in small dimension.
  - ◇ (Simpson's rule relies on formulae that are applicable or accurate only in dimenions say ≤ 10 or 12.
- For some special cases (like simple recourse), it might be possible to use numerical integration for your stochastic programming problem.

The Dartboard Method of Numerical Integration

- Throw darts at an area A. The percentage of darts that hit under the curve is like the integral.



Dartboard Integration

$$I(x,y) = \begin{cases} 1 & y \leq f(x) \\ 0 & y > f(x) \end{cases}$$

- Choose $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

$$\int_{x=a}^{b} f(x)dx \approx A \frac{\sum_{i=1}^{n} I(x_i, y_i)}{n}.$$

? How fast does it converge?

- This is more along the lines of what we will do.
- Stay tuned until next lecture (or maybe one after that)—Monte Carlo methods.

Bounds

- Since we in general can't explicity or numerically determine $\mathcal{Q}(x)$ or $\partial\mathcal{Q}(x)$, we will turn to methods to approximate this function and set.
- Methods will fall into two general categories
  - ◇ Methods with *known* error bounds
  - ◇ Methods with *statistical* error bounds (confidence intervals).
    - (This is the dartboard method).

Lower Bounds

- Suppose we are given a convex function f of a random variable ω. $f : \Omega \mapsto \Re$, where $\Omega \subseteq \Re^r$.
- $Q(\hat{x}, \omega) = \min_{y \in \Re^p_+} \{q^T y : Wy = h(\omega) - T(\omega)\hat{x}\}$

? For fixed $\hat{x}$, what do we know about the shape of $Q(\hat{x}, \omega)$?

Developing Lower Bounds

- For fixed $\hat{x}$, $Q(\hat{x}, \omega)$ is convex in ω!!!
- Why?
  - ◇ For the same reason as $Q(x, \hat{\omega})$ is a convex function of x for fixed $\hat{\omega}$.
  - ◇ It is the right-hand-side *value function* of a linear programming problem!
- Since $Q(\hat{x}, \omega)$ is convex, we will aim (first) to under-approximate the function by a linear function.
- We know we can do this becasue it is convex. In fact, we know the exact form of such an underestimating function...

Devloping Lower Bounds

- Choose some $\hat{\omega} \in \Omega$, and let $\eta \in \partial Q(\hat{x}, \hat{\omega})$. We know...

$$
\begin{aligned}
L(\hat{x}, \omega) &= Q(\hat{x}, \hat{\omega}) + \eta^T(\omega - \hat{\omega}) \leq Q(\hat{x}, \omega) \\
\mathbb{E}_\omega L(\hat{x}, \omega) &= \mathbb{E}_\omega \left[Q(\hat{x}, \hat{\omega}) + \eta^T(\omega - \hat{\omega})\right] \leq \mathbb{E}_\omega[Q(\hat{x}, \omega)] \\
\mathbb{E}_\omega L(\hat{x}, \omega) &= Q(\hat{x}, \hat{\omega}) + \eta^T(\mathbb{E}_\omega[\omega]) - \hat{\omega} \leq \mathbb{E}_\omega[Q(\hat{x}, \omega)] \\
\mathbb{E}_\omega L(\hat{x}, \omega) &= L(\mathbb{E}_\omega(\omega)) \leq \mathbb{E}_\omega[Q(\hat{x}, \omega)]
\end{aligned}
$$

⋆ That is, the expected lower bound is equivalent to evaluating the lower bound at the expected value.

Carrying On

- We would like the largest lower bound possible.
- The largest that $L(\mathbb{E}_\omega(\omega))$ can be is $Q(\hat{x}, \mathbb{E}_\omega[\omega])$, so we have shown that...

$$\mathbb{E}_\omega[Q(\hat{x}, \omega)] \geq Q(\hat{x}, \mathbb{E}_\omega[\omega])$$

- We get a *tight* lower bound on $\mathcal{Q}(\hat{x})$ by evauating $Q(\hat{x}, \bar{\omega})$.
- What did we *use* in the proof.
- This just relied on the fact that $Q(\hat{x}, \omega)$ was *convex* on Ω.

A General Theorem

- So, in general, if ϕ is a convex function ω of a random variable over its support Ω, then

$$\mathbb{E}_\omega \phi(\omega) \geq \phi(\mathbb{E}_\omega(\omega))$$

- Does this look familiar to anyone?
- What if I told you that it was called *Jensen's Inequality*
- What if I told you you proved this in a homework problem?

Look Familiar

Thm: Let $f : \Re^n \mapsto \Re$ be a convex function. Then for any $x_1, x_2, \ldots, x_k$, and all sets of "convex multipliers" $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\lambda_i \geq 0 \; \forall i = 1, 2, \ldots, k$,

$$f(\sum_{i=1}^k \lambda_i x_i) \leq \sum_{i=1}^k \lambda_i f(x_i).$$

- Probabilities are "convex multipliers".
- $f(\sum_{i=1}^k \lambda_i x_i) \approx \phi(\mathbb{E}_\omega(\omega))$
- $\sum_{i=1}^k \lambda_i f(x_i) \approx \mathbb{E}_\omega \phi(\omega)$

⋆ **Moral:** Maybe all that mathematics isn't a complete waste of time!

I Stand Corrected

- Well, maybe math *IS* a waste of time.
- Isn't what we have done obvious? What does it say?
- $Q(\hat{x}, \omega) = \min_{y \in \Re^p_+} \{q^T y : Wy = h(\omega) - T(\omega)\hat{x}\}$

$$\mathbb{E}_\omega Q(\hat{x}, \omega) \geq Q(\hat{x}, \mathbb{E}_\omega[\omega])$$

- ω affects the RHS.
- Minimizing a function considering ony one RHS in the constraints, you should be able to do "better" than if you consider many RHS's.

Our Only Example

minimize

$$x_1 + x_2$$

subject to

$$\begin{aligned} \omega_1 x_1 + x_2 &\geq 7 \\ \omega_2 x_1 + x_2 &\geq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

- $\omega_1 \sim \mathcal{U}[1, 4]$
- $\omega_2 \sim \mathcal{U}[1/3, 1]$

A Recourse Formulation

minimize

$$\mathcal{Q}(x_1, x_2) = x_1 + x_2 + 5 \int_{\omega_1=1}^{4} \int_{\omega_2=1/3}^{2/3} y_1(\omega_1, \omega_2) + y_2(\omega_1, \omega_2) d\omega_1 d\omega_2$$

subject to

$$\begin{aligned}
\omega_1 x_1 + x_2 + y_1(\omega_1, \omega_2) &\geq 7 \\
\omega_2 x_1 + x_2 + y_2(\omega_1, \omega_2) &\geq 4 \\
x_1 &\geq 0 \\
x_2 &\geq 0 \\
y_1(\omega_1, \omega_2) &\geq 0 \\
y_2(\omega_1, \omega_2) &\geq 0
\end{aligned}$$

AMPL — Be Afraid. Be Very Afraid

- Let's bound $\mathcal{Q}(2, 2)$.
- Class interactive portion. Improving our lower bound...

What Good Is This Stuff

- Big deal, so what if I know that a lower bound on $\mathcal{Q}(x)$
- The real trick is that you recursively partitioning the region Ω and the bounds become tighter and tighter.

Let $\mathcal{S} = \{\Omega^l, l = 1, 2, \ldots v\}$ be some partition of Ω. Do you believe me that

$$\mathbb{E}_\omega[Q(\hat{x}, \omega)] \geq \sum_{l=1}^{v} P(\omega \in \Omega^l) Q(\hat{x}, \mathbb{E}_\omega(\omega | \omega \in \Omega^l))$$

- Another on-the-fly AMPL example here...

Next Time

- Upper Bounds
- Using Bounds in Algorithms.
- HW#2 due
- Project Description Due
- No exceptions. No more Mr. Nice Guy.