IE 495 – Lecture 19

Stochastic Integer Programming

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Outline

- HW#2 Review
- Stochastic Integer Programming
- Properties of the recourse function
- Integer LShaped Method

HW#2

- Problem #1
  - ◇ Doesn't depend on $\mathbb{E}[\xi] < \infty$!
  - ◇ Complete Recourse Definition...
  - ◇ Case by case...
- Problem #2—Ugly Math
  - ◇ Ph.D. students—Get used to it!
  - ◇ Masters students—I apologize
- Problem #3
  - ◇ *No one* formulated SMPS correctly.
  - ◇ We'll go over a bit in class...

If the World Ended Today...

- Based on the first number in the box on your homework...

Score	Grade	#
≥ 18	A	9
≥ 17.7	A-	4
≥ 17.4	B+	1
≥ 16.8	B	0
< 16.8	B-	2

(Mixed) Integer Programming

- Linear programming where some of the variables are constrained to take ony integer values

$$\min_{x \in X} \{c^T x | Ax = b\}$$

- $X = \{x \in \mathbb{Z}_+^{n_1} \times \Re_+^{n-n_1}\}$
- If $n = n_1$, *(pure) integer programming,*
- Otherwise *mixed integer programming*

Why MIP

- "Natural" integrality of decision variables.
  - Depending on scale of integer variables, often a linear approximation is close enough.
- Yes/No decisions
- Logical conditions

Stochastic MIP

- Given random outcome ω and set $X = \{x \in \mathbb{Z}_+^{n_1} \times \Re_+^{n-n_1}\}$, we want to

$$\min_{x \in X} \{c^T x | Ax = b, T(\omega)x = h(\omega)\}.$$

- If we must decide on x before the outcome ω is known, we need to do something.
- We will do our favorite thing, and equip the problem with *recourse*:
- $q \in \Re^p$: Recourse costs
- $W \in \Re^{m \times p}$: Recourse matrix
- $Y = \{y \in \mathbb{Z}_+^{p_1} \times \Re_+^{p-p_1}\}$

Stochastic MIP

$$\min_{x \in X : Ax = b} \left\{ c^T x + \mathbb{E}_\omega \left[\min_{y \in Y} \{ q^T y : W y = h(\omega) - T(\omega) x \} \right] \right\}$$

- *Second stage value function*, or *recourse (penalty) function* $v : \Re^m \mapsto \Re$.
  - $v(z) \equiv \min_{y \in Y} \{ q^T y : W y = z \}$,
  - For any vector z of "deviations in the random constraints $T(\omega) x = h(\omega)$", it describes the corresponding cost.
- *Expected Value Function*, or *Expected minimium recourse function* $\mathcal{Q} : \Re^n \mapsto \Re$.
  - $\mathcal{Q}(x) \equiv \mathbb{E}_\omega [v(h(\omega) - T(\omega) x)]$
  - For any policy $x \in \Re^n$, it describes the expected cost of the recourse.

Stochastic MIP

- Recall that if Ω was finite, we could write the (deterministic equivalent) of a stochastic LP
  - ◇ Just a large scale LP
- We can do the same for stochastic MIP
  - Just a large-scale IP
  - ◇ But a large-scale IP with a very weak linear programming relaxation $\Rightarrow$ not likely to be solved by "off-the-shelf" software like cplex.

Nasty, Nasty, Functions

- If you recall, our L-Shaped method for stochastic LP was based on knowing "nice" properties of the second stage value function ($v(z)$) or the Expected Value Function $\mathcal{Q}(x)$.

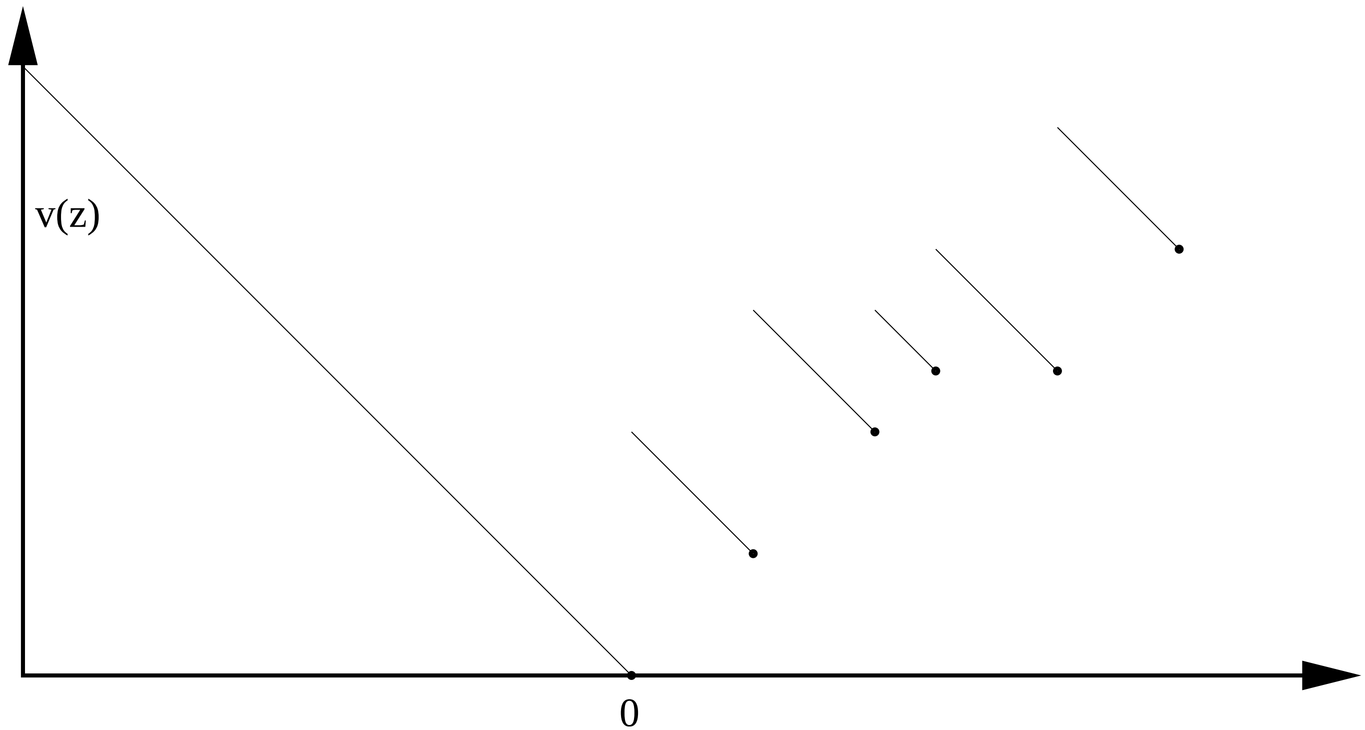
⋆ We'll study it again!

$$v(z) = \min_{y \in Y} \{q^T y | Wy = z\}, z \in \Re^m$$

- Here are two properties...
  - ◇ $v(z)$ is lower semicontinuous on $\Re^m$
  - ◇ The discontinuity points of v are contained in a countable union of hyperplanes in $\Re^m$

$v(z)$

$$v(z) = \min_{y \in Y} \{2y_1 + 5y_2 + 6y_3 + y_4 | 2y_1 + 5y_2 + ty_3 - y_4 = z\}, Y = \{\mathbb{Z}_+^3 \times \Re_+\}$$



Lower Semicontinuous?

- $f : D \mapsto \Re$ is *lower semicontinuous* at $\hat{x}$ if $\forall \epsilon > 0 \exists \delta > 0$ such that $x \in S, \|x - \hat{x}\| < \delta \Rightarrow f(x) - f(\hat{x}) > -\epsilon$.
- $f : D \mapsto \Re$ is *lower semicontinuous* at $\hat{x}$ if for any sequence $\{x_n\} \rightarrow \hat{x}$, with $\{f(x_n)\} \rightarrow \hat{f}$, $\hat{f} \geq f(\hat{x})$.

Expected Recourse Function

- Now we will consider properties of $\mathcal{Q}(x) = \mathbb{E}_\omega[v(h(\omega) - T(\omega)x)], x \in \Re^n$
- Let $D(x) = \{\omega \in \Omega | v$ is not continuous at $h(\omega) - T(\omega)x\}$
- $\mathcal{Q}(x)$ is a finite lower semicontinuous function on $\Re^n$.
- $\mathcal{Q}$ is continuous at $\hat{x}$ if $P(\omega \in D(\hat{x})) = 0$
  - ◇ The sum of lower semicontinuous functions is lower semicontinuous
  - ◇ The condition $P(\omega \in D(\hat{x})) = 0$ implies that the set of ω such that v is discontinuous at $h(\omega) - T(\omega)x$ is negligible in the integral $\mathbb{E}_\omega[v(h(\omega) - T(\omega)x)]$.
  - ◇ Then continuity of Q follows

Algorithms for Stochastic IP

- I don't want to steal people's thunder, but I will *briefly* discuss the following methods.
- Integer L-Shaped method
- Dual Decomposition Method
- Stochastic Branch-and-Bound
- Structured Enumeration

Stochastic IP—The Simple Case

$$\min_{x \in X : Ax = b} \left\{ c^T x + \mathbb{E}_\omega \left[\min_{y \in Y} \{ q^T y : Wy = h(\omega) - T(\omega) x \} \right] \right\}$$

- Suppose $X \subseteq \mathbb{Z}^{\bar{n}} \times \Re_+^{n - \bar{n}}$
- $Y \subseteq \Re_+^p$.
- What is the shape of $\mathcal{Q}(x)$?
  - It is *convex!*
- Do the L-shaped method except that you solve an *integer* program as the master problem.

Integer L-Shaped Method

Gilbert Laporte and François Louveaux, "The integer L-shaped method for stochastic integer programs with complete recourse", Operations Research Letters, 13:133-142, 1993.

Designed to work on

$$\min_{x \in X : Ax = b} \{c^T x + \mathcal{Q}(x)\}$$

$$\mathcal{Q}(x) = \mathbb{E}_\omega \left[\min_{y \in Y} \{q^T y : Wy = h(\omega) - T(\omega)x\} \right]$$

- $X \subseteq \{0,1\}^n$
- $Y \subseteq \mathbb{Z}_+^{\bar{p}} \times \Re_+^{p-\bar{p}}$

New Optimality Cuts

For $x^k \in X \subseteq \{0,1\}^n$, define the set

$$S^k = \{j | x_j^k = 1\}.$$

Thm: The cut

$$\theta \geq (\mathcal{Q}(x^k) - L)\left(\sum_{j \in S^k} x_j - \sum_{j \notin S^k} x_j - |S^k| + 1\right) + L.$$

is a valid optimality cut for $\mathcal{Q}(x)$

Next Time

- Finish all of Stochastic IP, including example of Integer L-Shaped method.
- I'll try to figure out what to do about the project presentations...