IE 495 – Lecture 2

Stochastic Programming Modeling

Prof. Jeff Linderoth

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Grading Policy

I forgot to add my grading policy for this semester on the syllabus...

- I will use a “sampling-based” grading scheme.
- For the assigned problems, I will grade one (maybe two) problems in-depth.
  - ◇ These problems will be out of 7 points.
  - ◇ The remaining problems will be worth 3 points.

⋆ I will always produce a full set of solutions.

Bueller? Bueller? Anyone?

Survey results...

- Happy: Most people want to do a project.
  - Some people even want to do both!
- Sad: Less than half have taken Nonlinear Programming.
  - That's OK, we'll introduce/review as needed
- Some people don't want much theory. Tough Toenails!

Today's Outline

- Review
- Stages and Decisions in Stochastic Programs
  - Wait-and-see vs. Here-and-now
- Dealing with Randomness in Linear Programs
  - Guess
    - Risk aversion
  - Chance constraints
  - Penalize shortages
  - Recourse actions
- Farmer Ted – A recourse problem

Please don't call on me!

- What does the term *programming* mean in stochastic programming?
- What is the expected value of (positive-valued) discrete random variable ξ?
- What is a probability space?
  - Do you *care* what a probability space is?

A Random Linear Program

minimize

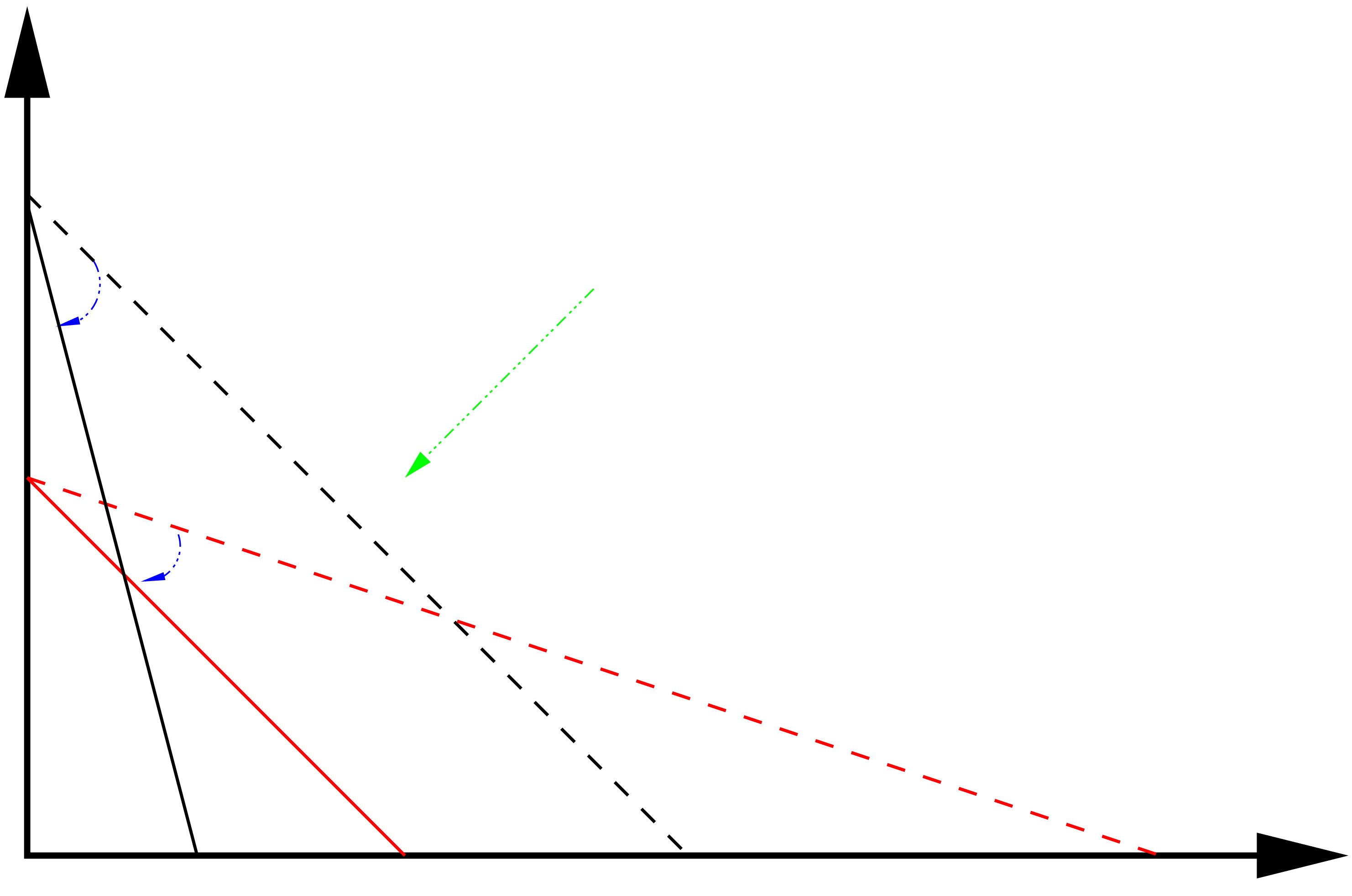
$$x_1 + x_2$$

subject to

$$\begin{aligned} \omega_1 x_1 + x_2 &\geq 7 \\ \omega_2 x_1 + x_2 &\geq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

- $\omega_1 \sim \mathcal{U}[1, 4]$
- $\omega_2 \sim \mathcal{U}[1/3, 1]$

Worth 1000 Words?



What To Do?

- How do we solve this problem?
- What do you *mean* by solving this problem?
- Suppose it is possible to decide about x *after* the observation of the random vector ω?
  - ◇ We can interpret this as a *wait-and-see* approach
- Can we solve the problem then?
  - ◇ I sure the heck hope so – it's just a simple deterministic linear program!

Here and Now

- Generally, "wait-and-see" is not an appropriate model of how things work.

⇒ We need to decide on x *before* knowing the values of ω.

- In order for the problem to make sense in this case, we need to decide what to do about not knowing ω_1, ω_2.
- Three suggestions
  - ◇ Guess at uncertainty
  - ◇ Probabilistic Constraints
  - ◇ *Penalize Shortfall*

Guess Away!

- We will guess reasonable values for ω_1, ω_2
  - Like I mentioned last lecture, this is what people normally do.
  - What should be guess?
- I will offer three (reasonable) suggestions – each of which tells us something about our level of "risk"
  - *Unbiased*: Choose mean values for each random variable
  - *Pessimistic*: Choose worst case values for ω
  - *Optimistic*: Choose best case values for ω

Unbiased

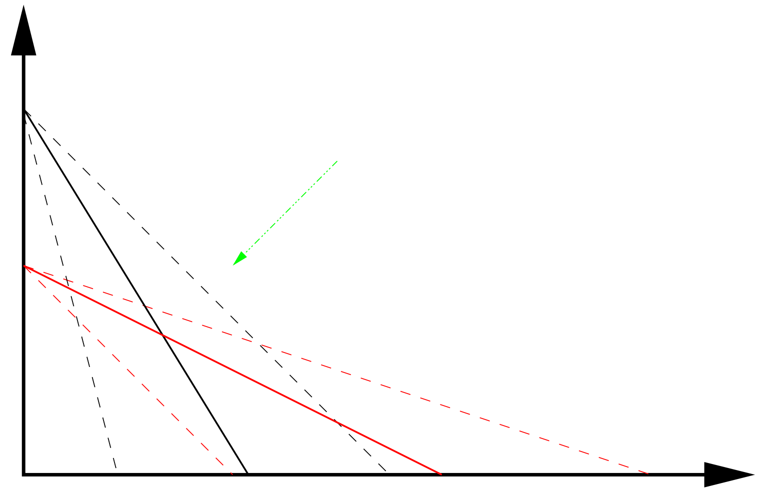
- $\hat{\omega} \equiv \mathbb{E}(\omega) = (5/2, 3/2)$

minimize

$$x_1 + x_2$$

subject to

$$\begin{aligned} \frac{5}{2}x_1 + x_2 &\geq 7 \\ \frac{3}{2}x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- $\hat{v} = 50/11$
- $(\hat{x}_1, \hat{x}_2) = (18/11, 32/11)$

Pessimistic

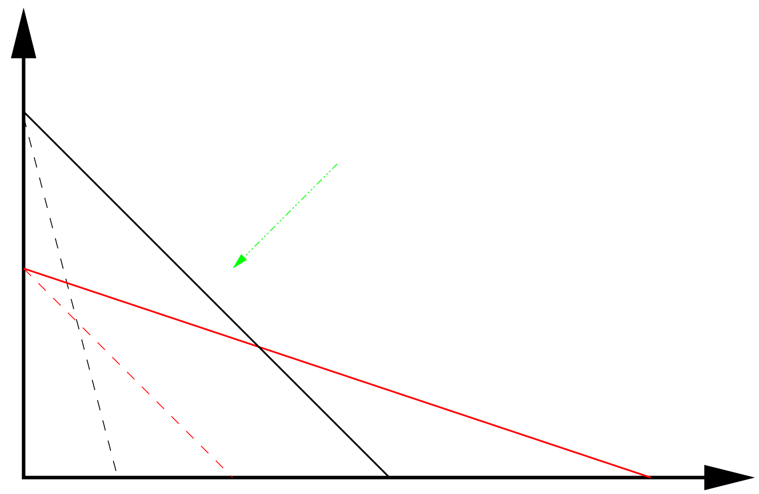
- $\hat{\omega} = (1, 1/3)$

minimize

$$x_1 + x_2$$

subject to

$$\begin{aligned} 1x_1 + x_2 &\geq 7 \\ 1/3x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Picture...

- $\hat{v} = 7$
- $(\hat{x}_1, \hat{x}_2) = (0, 7)$

Optimistic

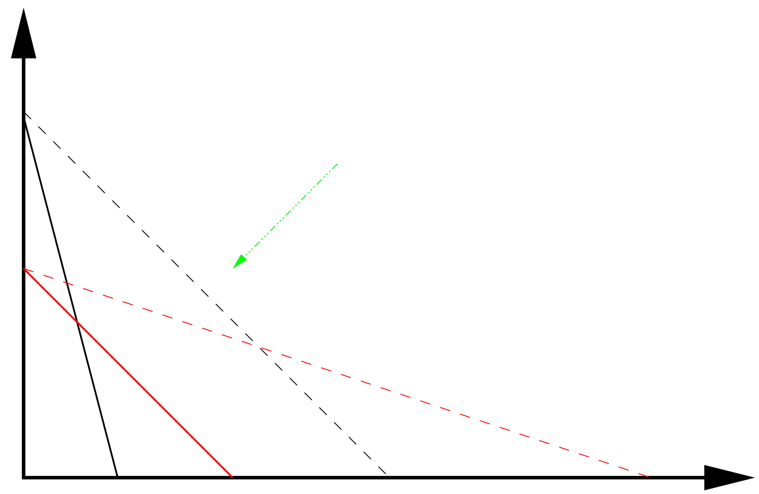
- $\hat{\omega} = (4, 1)$

minimize

$$x_1 + x_2$$

subject to

$$\begin{aligned} 4x_1 + x_2 &\geq 7 \\ 1x_1 + x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$



- $\hat{v} = 4$
- $(\hat{x}_1, \hat{x}_2) = (4, 0)$

Pros and Cons

+ Easy!
  - ◇ Solve a deterministic problem oif the same size as the original random problem
+ Only "rough" information about the randomness ω is needed.
– Only takes into account one "case" of what the randomness might be
– There might even be ω for which the chosen x is infeasible.

Chance Constrained

- Another (probably more reasonable) approach. Let's enforce that the *probability* of a constraint holding is sufficiently large.

Let's add the constraints

$$
\begin{aligned}
\mathrm{P}\{\omega_1 x_1 + x_2 \geq 7\} &\geq \alpha_1 \\
\mathrm{P}\{\omega_2 x_1 + x_2 \geq 4\} &\geq \alpha_2
\end{aligned}
$$

Or maybe the constraint

$$\mathrm{P}\{\omega_1 x_1 + x_2 \geq 7, \omega_2 x_1 + x_2 \geq 4\} \geq \alpha$$

Chance Constraints

- Note for $\alpha_1, alpha_2, \alpha = 1$ this is equivalent to a normal (deterministic) problem

? How do we solve probabilistically constrained problems?

  – It's (very) difficult

  ◇ Stay tuned.

    ⇒ We will learn (a little) bit about these problems later in the course

Approach III – Penalize Shortfall

- We will accept infeasibility, but penalize the expected shortage.
- Notation:
  - $x^+ \equiv \max(0, z)$: The positive part of z.
  - $x^- \equiv \max(0, -z)$: The negative part of z.
- Then, for the constraint $\omega_1 x_1 + x_2 \geq 7$, the shortfall is $(\omega_1 x_1 + x_2 \geq 7)^-$
- For each constraint, assign (unit) shortfall costs q_1, q_2.
- Optimization problem becomes...

$$\min_{x \in \Re^2_+} \{x_1 + x_2 + q_1 \mathbb{E}_{\omega_1} \left[(\omega_1 x_1 + x_2 - 7)^-\right] + q_2 \mathbb{E}_{\omega_2} \left[(\omega_2 x_1 + x_2 - 4)^-\right]\}$$

Yikes!

- Yes, I concur that the function we are trying to optimize looks ugly.
- However, it is convex.
  - ◇ You will learn this formally later. (Yuck! Theory!)
- In fact, it is not too hard to see that the problem is equivalent to the following:

$$\min_{x \in \Re^2_+} \left\{ x_1 + x_2 + \mathbb{E}_\omega \left[\min_{y \in \Re^2_+} \left\{ q_1 y_1 + q_2 y_2 : \begin{array}{c} \omega_1 x_1 + x_2 + y_1 \geq 7 \\ \omega_2 x_1 + x_2 + y_2 \geq 4 \end{array} \right\} \right] \right\}$$

Recourse Function

- Let's write the problem in terms of x only

$$\min_{x \in \Re^2_+} \{x_1 + x_2 + \mathcal{Q}(x_1, x_2)\}$$

where

$$\mathcal{Q}(x_1, x_2) = \mathbb{E}_\omega \left[\min_{y \in \Re^2_+} \left\{ q_1 y_1 + q_2 y_2 : \begin{array}{c} y_1 \geq 7 - \omega_1 x_1 - x_2 \\ y_2 \geq 4 - \omega_2 x_1 - x_2 \end{array} \right\} \right]$$

- $\mathcal{Q}(x_1, x_2)$ is called the *recourse function.*
- For a given decision x_1, x_2, what do we do (recourse)?
- In this case, it is simply to penalize the shortfall.
- y_1, y_2 will be exactly the shortfall in constraints 1 and 2.

Decisions, Stages, and Recourse

When dealing with "here-and-now" decision problems, in general, we don't have to necessarily penalize shortfall, but we might be able to take "corrective action" – *recourse*!

Consider a planning problem with two periods. The following sequence of events occurs.

1. We make a decision now (first-period decision)
2. Nature makes a random decision ("stuff" happens)
3. We make a second period decision that attempts to repair the havoc wrought by nature in (2). (recourse)

Recourse Example – Farmer Ted

- Farmer Ted can grow Wheat, Corn, or Beans on his 500 acres.
- Farmer Ted requires 200 tons of wheat and 240 tons of corn to feed his cattle
  - These can be grown on his land or bought from a wholesaler.
  - Any production in excess of these amounts can be sold for \$170/ton (wheat) and \$150/ton (corn)
  - Any shortfall must be bought from the wholesaler at a cost of \$238/ton (wheat) and \$210/ton (corn).
- Farmer Ted can also grow beans
  - Beans sell at \$36/ton for the first 6000 tons
  - Due to economic quotas on beet production, beans in excess of 6000 tons can only be sold at \$10/ton

The Data

- 500 acres available for planting

	Wheat	Corn	Beans
Yield (T/acre)	2.5	3	20
Planting Cost ($/acre)	150	230	260
Selling Price	170	150	36 ($\leq$ 6000T)
			10 ($>$6000T)
Purchase Price	238	210	N/A
Minimum Requirement	200	240	N/A

Formulate the LP – Decision Variables

- $x_{W,C,B}$ Acres of Wheat, Corn, Beans Planted
- $w_{W,C,B}$ Tons of Wheat, Corn, Beans sold (at favorable price).
- e_B Tons of beans sold at lower price
- $y_{W,C}$ Tons of Wheat, Corn purchased.

⋆ Note that Farmer Ted has *recourse*. After he observes the weather event, he can decide how much of each crop to sell or purchase!

- (Farmer Fred from lecture #1 had no recourse – his recourse action was to simply count the profits).

Formulation

maximize

$$-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B$$

subject to

$$\begin{aligned}
x_W + x_C + x_B &\leq 500 \\
2.5x_W + y_W - w_W &= 200 \\
3x_C + y_C - w_C &= 240 \\
20x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{aligned}$$

Solution with (expected) yields

	Wheat	Corn	Beans
Plant (acres)	120	80	300
Production	300	240	6000
Sales	100	0	6000
Purchase	0	0	0

- Profit: $118,600

Planting Intuition

- Farmer Ted is happy to see that the LP solution corresponds to his intuition.
  - ◇ Plant the land necessary to grow up to his quota limit of beans.
  - ◇ Plant land necessary to meet his requirements for wheat and corn
  - ◇ Plant remaining land with wheat – sell excess.

It's the Weather, Stupid!

- Farmer Ted knows well enough to know that his yields aren't always precisely $Y = (2.5, 3, 20)$. He decides to run two more scenarios
- Good weather: $1.2Y$
- Bad weather: $0.8Y$

Formulation – Good yields

maximize

$$-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B$$

subject to

$$\begin{aligned}
x_W + x_C + x_B &\leq 500 \\
3x_W + y_W - w_W &= 200 \\
3.6x_C + y_C - w_C &= 240 \\
24x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{aligned}$$

Solution with good yields

	Wheat	Corn	Beans
Plant (acres)	183.33	66.67	250
Production	550	240	6000
Sales	350	0	6000
Purchase	0	0	0

- Profit: \$167,667

Formulation – Bad Yields

maximize

$$-150x_W - 230x_C - 260x_B - 238y_W + 170w_W - 210y_C + 150y_C + 36w_B + 10e_B$$

subject to

$$\begin{aligned}
x_W + x_C + x_B &\leq 500 \\
2x_W + y_W - w_W &= 200 \\
2.4x_C + y_C - w_C &= 240 \\
16x_B - w_B - e_B &= 0 \\
w_B &\leq 6000 \\
x_W, x_C, x_B, y_W, y_C, e_B, w_W, w_C, w_B &\geq 0
\end{aligned}$$

Solution – Bad Yields

	Wheat	Corn	Beans
Plant (acres)	100	25	375
Production	200	60	6000
Sales	0	0	6000
Purchase	0	180	0

- Profit: \$59,950

What to do?

- Obviously the answer is quite dependent on the weather and the respective yields.
- Another main issue is on bean production. Without knowing the weather/yield, he can't determine the proper amount of beans to plant to maximize his quota and not have to sell any at the unfavorable price.
- It's impossible to make a perfect decision, since planting decisions must be made now, but purchase and sales decisions can be made later.

Maximize *Expected* Profit

- Assume that the three scenarios occur with equal proability.
- Attach a scenario subscript $s = 1, 2, 3$ to each of the purchase and sale variables.
  - 1: Good, 2: Average, 3: Bad

Ex. w_{C2} : Tons of corn sold at favorable price in scenario 2

Ex. e_{B3} : Tons of beans sold at unfavorable price in scenario 3.

Expected Profit

- An expression for Farmer Ted's Expected Profit is the following:

$$
\begin{aligned}
&150x_W - 230x_C - 260x_B \\
&+1/3(-238y_{W1} + 170w_{W1} - 210y_{C1} + 150y_{C1} + 36w_{B1} + 10e_{B1}) \\
&+1/3(-238y_{W2} + 170w_{W2} - 210y_{C2} + 150y_{C2} + 36w_{B2} + 10e_{B2}) \\
&+1/3(-238y_{W3} + 170w_{W3} - 210y_{C3} + 150y_{C3} + 36w_{B3} + 10e_{B3})
\end{aligned}
$$

Expected Value Problem – Constraints

$$
\begin{aligned}
x_W + x_C + x_B &\leq 500 \\
3x_W + y_{W1} - w_{W1} &= 200 \\
2.5x_W + y_{W2} - w_{W2} &= 200 \\
2x_W + y_{W3} - w_{W3} &= 200 \\
3.6x_C + y_{C1} - w_{C1} &= 240 \\
3x_C + y_{C2} - w_{C2} &= 240 \\
2.4x_C + y_{C3} - w_{C3} &= 240 \\
24x_B - w_{B1} - e_{B1} &= 0 \\
20x_B - w_{B2} - e_{B2} &= 0 \\
16x_B - w_{B3} - e_{B3} &= 0 \\
w_{B1}, w_{B2}, w_{B3} &\leq 6000 \\
\text{All vars} &\geq 0
\end{aligned}
$$

Optimal Solution

	Wheat	Corn	Beans	
s	Plant (acres)	170	80	250
1	Production	510	288	6000
1	Sales	310	48	6000
1	Purchase	0	0	0
2	Production	425	240	5000
2	Sales	225	0	5000
2	Purchase	0	0	0
3	Production	340	192	4000
3	Sales	140	0	4000
3	Purchase	0	48	0

- (Expected) Profit: $108,390

Solution Characteristics

- Best solution allocates land for beans to always avoid having to sell them at the unfavorable price.
- Corn is planted so that the requirement is met in the average scenario.
- The remaining land is allocated to wheat.

⋆ Again, it is impossible to find a solution that is ideal under all circumstances. Decisions in stochastic models are balanced, or *hedged* against the various scenarios.

AMPL

Fortune Tellers

- Suppose Farmer Ted could *with certainty* tell whether or not the upcoming growing season was going to have good yields, average yields, or bad yields.
  - ◇ His bursitits was acting up
  - ◇ Consulting the Farmer's Almanac
  - ◇ Hire a fortune teller
- The real point here is how *much* Farmer Fred would be willing to pay for this "perfect" information.

⋆ In real-life problems, how much is it "worth" to invest in better (or perfect) forecasting technology?

What's it worth?

- If $p = 0.5$ – i.e. half of the seasons are wet, and half of the seasons are dry, how much more money could he make?
- In the wet seasons, he would plant all corn and make \$100.
- In the dry seasons, he would plant all wheat and make \$40.
- In the long run, his profit would be $0.5(100) + 0.5(40) = \$70$.
- Constrast this to the optimal (in the presence of uncertainty) profit of planting all beans : \$57.5.
- We can this difference (\$70 - \$57.5) the *expected value of perfect information*(EVPI)

What's it worth?

- With perfect information, Farmer Ted's would plant (wheat, corn, beans).
  - Good yield: (183.33, 66.67, 250), Profit: \$167,667
  - Average yield: (120, 80, 300), Profit: \$118,600
  - Bad yield: (100, 25, 375), Profit: \$59,950
- Assuming each of these scenarios occurs with probability 1/3, his long run average profit would be
  - $(1/3)(167667) + (1/3)(118600) + (1/3)(59950) = 115406$
- With his (optimal) "here-and-now" decision of (170, 80, 250), he would make a long run profit of 108390
- This difference (115406-108390) is the *expected value of perfect information*(EVPI)

Readings

- 1.2, 2.1, 2.2, 2.3, 2.4, 2.7
- If you want to review some math – 2.9