IE 495 – Lecture 20

Stochastic Integer Programming

Prof. Jeff Linderoth

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Outline

- Stochastic Integer Programming
  - Integer LShaped Method
  - Dual Decomposition Method
  - Stochastic Branch and Bound
  - Structured Enumeration

Stochastic MIP

$$\min_{x \in X : Ax = b} \left\{ c^T x + \mathbb{E}_\omega \left[\min_{y \in Y} \{ q^T y : Wy = h(\omega) - T(\omega)x \} \right] \right\}$$

- $X = \{x \in \mathbb{Z}_+^{n_1} \times \Re_+^{n-n_1}\}$
- $Y = \{y \in \mathbb{Z}_+^{p_1} \times \Re_+^{p-p_1}\}$
  - ◇ $v(z) \equiv \min_{y \in Y} \{q^T y : Wy = z\}$,
  - ◇ $\mathcal{Q}(x) \equiv \mathbb{E}_\omega [v(h(\omega) - T(\omega)x)]$

$$\min_{x \in X : Ax = b} \left\{ c^T x + \mathcal{Q}(x) \right\}.$$

Stochastic MIP

- Recall that if Ω was finite, we could write the (deterministic equivalent) of a stochastic LP
  - ◇ Just a large scale LP
- We can do the same for stochastic MIP
  - Just a large-scale IP
  - ◇ But a large-scale IP with a very weak linear programming relaxation $\Rightarrow$ not likely to be solved by "off-the-shelf" software like cplex.

Stochastic IP—The Simple Case

$$\min_{x \in X : Ax = b} \left\{ c^T x + \mathbb{E}_\omega \left[\min_{y \in Y} \{ q^T y : Wy = h(\omega) - T(\omega) x \} \right] \right\}$$

- Suppose $X \subseteq \mathbb{Z}^{\bar{n}} \times \Re_+^{n - \bar{n}}$
- $Y \subseteq \Re_+^p$.
- Integrality *only* in the first stage?
- What is the shape of $\mathcal{Q}(x)$?
  - ◇ It is *convex!*

⋆ Idea to solve—Do the L-shaped method except that you solve an *integer* program as the master problem.

This case is *almost* as easy as two-stage LP w/recourse

Integer L-Shaped Method

Gilbert Laporte and François Louveaux, "The integer L-shaped method for stochastic integer programs with complete recourse", **Operations Research Letters**, 13:133-142, 1993.

Designed to work on

$$\min_{x \in X : Ax = b} \{c^T x + \mathcal{Q}(x)\}$$

$$\mathcal{Q}(x) = \mathbb{E}_\omega \left[\min_{y \in Y} \{q^T y : Wy = h(\omega) - T(\omega)x\} \right]$$

- $X \subseteq \{0,1\}^n$
- $Y \subseteq \mathbb{Z}_+^{\bar{p}} \times \Re_+^{p-\bar{p}}$
- *All binary* first stage variables, arbitrary second stage

New Optimality Cuts

For $x^k \in X \subseteq \{0,1\}^n$, define the set

$$S^k = \{j | x_j^k = 1\}.$$

Thm: The cut

$$\theta \geq (\mathcal{Q}(x^k) - L) \left(\sum_{j \in S^k} x_j - \sum_{j \notin S^k} x_j - |S^k| + 1 \right) + L.$$

is a valid optimality cut for $\mathcal{Q}(x)$

- "Valid Optimality Cut" means
  - ◇ Cut is tight at x
  - ◇ Inequality holds for all feasible x.

Proof

$$\theta \geq (\mathcal{Q}(x^k) - L)\left(\sum_{j \in S^k} x_j - \sum_{j \notin S^k} x_j - |S^k| + 1\right) + L.$$

- Consider the quantity $A \equiv \sum_{j \in S^k} x_j - \sum_{j \notin S^k} x_j$. $A \leq |S^k|$. $A = |S^k|$ if and only if S^k is the set based on $x = (x_1, x_2, \ldots x_n)$.
- If $A = |S^k|$, then the cut is $\theta \geq \mathcal{Q}(x^k)$.
- If $A < |S^k|$, then (x) is *not* the solution on which S^k is based. In this case,
  - ◇ $A < |S^k| \Rightarrow A \leq |S^k| - 1$

Proof, cont.

$A \leq |S^k| - 1 \ldots$

$\Rightarrow \left(\sum_{j \in S^k} x_j - \sum_{j \notin S^k} x_j - |S^k| + 1\right) \leq 0$

$\Rightarrow (\mathcal{Q}(x^k) - L)\left(\sum_{j \in S^k} x_j - \sum_{j \notin S^k} x_j - |S^k| + 1\right) \leq 0,$

$\Rightarrow$ so thw cut is $\theta \geq L + M$, with $M \leq 0$, and the cut must be valid for this x.

QUITE ENOUGH DONE.

Integer L-Shaped Method—Algorithm

- We just will do a "single cut" version of the Integer L-Sshaped method. (Multicut is a simple extension).

0. Let $v = 0$, let $\bar{z} = \infty$
1. Select node. If none exist, stop.
2. $v = v + 1$. Solve problem, if infeasible, fathom node and go to 1. Let (x^v, θ^v) be solution to problem at node v. If $c^T x^v + \theta^v > \bar{z}$, fathom node and go to 1.
3. If x violates integrality, *branch*, creating new nodes. Go to 1.
4. Compute $\mathcal{Q}(x^v)$. If $z^v = c^t x^v + \mathcal{Q}(x^v) < \bar{z}$, then update $\bar{z} \leftarrow z^v$.
5. If $\theta^v \geq \mathcal{Q}(x^v)$, fathom the node and go to 1. Otherwise, add (integer) optimality cut, and go to 2.

Dual Decomposition

C. C. Carøe and R. Schultz, "Dual Decomposition in Stochastic Integer Programming," **Operations Research Letters**, 24:37-45, 1999.

- Main idea, having to choose only one solution x for each scenario s is a bummer.
  - ◇ If we didn't have this restriction, the problem would be "easy" in the sense that we could just solve each scenario independently.

Dual Decomposition

minimize

$$\sum_{s \in S} p_s c^T x_s + q^T y_s$$

subject to

$$\begin{aligned} Ax &= b \\ T_s x + W y_s &= h_s && \forall s \in S \\ x_s &\in X && \forall s \in S \\ y_s &\in Y && \forall s \in S \\ x_1 = x_2 = & \ldots && = x_s \end{aligned}$$

Relax Nonanticipativity

- The constraints $x_1 = x_2 = \ldots = x_s$ are like nonanticipativity constraints.
- We can write the equalities as

$$\sum_{s \in S} H_s x_s = 0.$$

Dual Decomposition

$$
\begin{aligned}
x_1 &= \sum_{s \in S} p_s x_s \\
x_2 &= \sum_{s \in S} p_s x_s \\
&\vdots \\
x_s &= \sum_{s \in S} p_s x_s
\end{aligned}
$$

- Use Lagrangian relaxation. Provides a lower bound on the optimal solution.
- If solution is integer feasible, then it is optimal. Otherwise *branch*.
- More details in a presentation.

Stochastic Branch and Bound

V. I. Norkin and Y. M. Ermoliev and A. Ruszczyński, "On Optimal Allocation of Indivisibles under Uncertainty," **Operations Research**, 46:381-392, 1998.

- Shares many similarities to the Monte Carlo Approach
- Very general
  - ◇ Works on (mixed) integer variable in both stages.
  - ◇ Arbitrary probability distributions for random parameters

Stochastic Branch and Bound

$$z^*(X) \min_{x \in X} F(x)$$

- In this description, we assume X is a finite set.
- Partition feasible set X into partition $\mathcal{P}^k \equiv \{X^1, X^2, \ldots X^{n(k)}\}$ at iteration k
- $z^*(X) = \min_{j=1}^{n(k)} z^*(X^j)$
- In general, we assume that we can compute (maybe statistical) upper and lower bounds $L(X^j) \leq z^*(X^j) \leq U(x^j)$

Stochastic Branch and Bound

1. Select "Record Set" $\bar{X} \in \arg\min_j \{L(X^j)\}$. Select "Approximate Solution" $x^k \in \arg\min_j \{U(x^j)\}$.
2. If $\bar{X}$ is not a singleton, partition $\bar{X}$ and update the working parition accordingly.
3. Update (estimates) of $L(X^j)$ and $U(X^j)$, paying most attention to (subsets of) the record set.
4. Remove subsets of X^j that contain no feasible solutions. Bounding out can only be applied if the estimates of the bounds are exact.
5. Go to 1 unless a stopping criteria is met.

Stochastic Branch and Bound

- A typical stopping criterion is that there is some (singleton) record set $\hat{X}$ where $U(\hat{X}) < L(X^k)$ for all other subsets k.

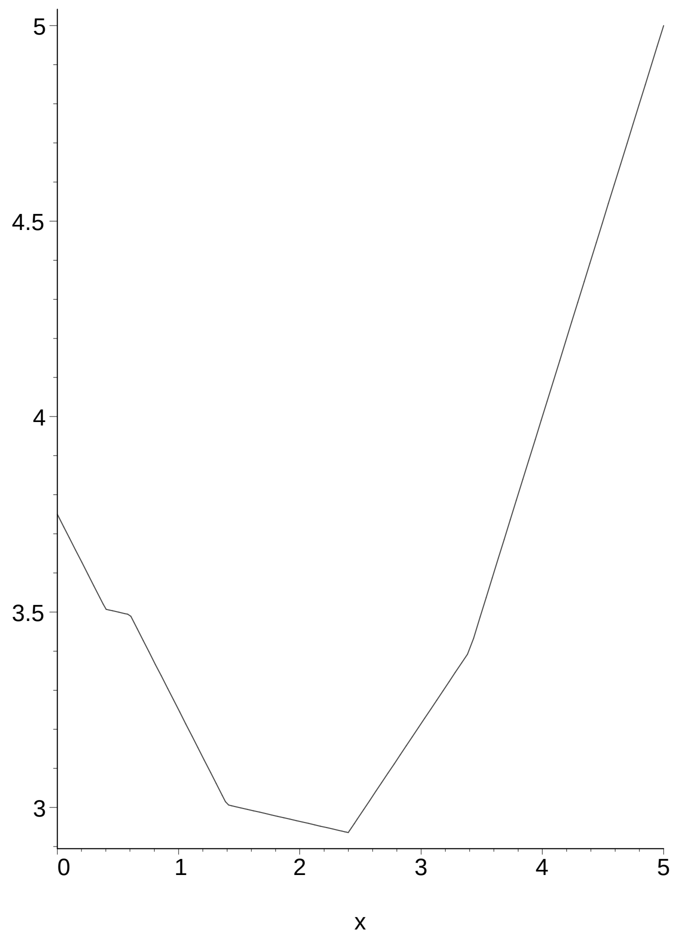
Stochastic Branch and Bound–Example

$$\min\{x + \mathcal{Q}(x) | x \in \{0, 1, \ldots, 10\}$$

- $\mathcal{Q}(x) = \mathbb{E}_\omega[v(x, \omega)]$
- $v(x, \omega) = \min\{1.5y : y \geq \omega - x, y \in \mathbb{Z}_+ = 1.5\lceil \omega - x \rceil^+$

Your notes here

$$x + \mathcal{Q}(x)$$



Projects

- Everyone will prepare a short report. No more than 4 pages.
  - Background
  - Goal of project
  - Things I Learned
  - Conclusions
- 15 minute (NO MORE THAN 15 MINUTE) presentation to class.
  - Six people present on 4/23
    - Any volunteers? Otherwise, I am going to "volunteer" six people.
  - Everyone else present on 5/1 (when our final is scheduled).

Final Exam

- I will pass out final on 4/23.
- It will be due on 5/1.
- It will be of comparable difficulty to the homeworks.
  - It will require some modeling.
  - It will require some sampling/computational component.

Next Time...

- Multistage Stochastic Programming
  - Modeling Issues
- Nested Bender's Decomposition