IE 495 – Lecture 21

Multistage Stochastic Programming

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Outline

- HW Fixes
- Multistage Stochastic Programming
  - Modeling
  - Nested Decomposition Method

Homework Fixes

- Problem 2...
  - $\Omega^I = \{\omega_1 \times \omega_2 | 1 \leq \omega_1 \leq 5/2, 1/3 \leq \omega_2 \leq 2/3\}$
  - $\Omega^{II} = \{\omega_1 \times \omega_2 | 5/2 \leq \omega_1 \leq 4, 1/3 \leq \omega_2 \leq 2/3\}$
- Problem 3...
  - The constraints with ω should be *equality* constraints
- Problem 4...
  - Should be a *maximization* problem

Grade Data Collection

Stochastic Integer Programming (Please Don't Call On Me!)

- How would you solve a really small instance?
- How would you solve an instance with integer variables only in the first stage?
- What is an "optimality cut" in the context of the integer L-Shaped Method?
- For what problem are the optimality cuts we showed last time valid?
- Name one manner in which we might obtain lower and upper bounds to use in the stochastic branch and bound method

Jacob and MIT

- We are given a universe N of investment decisions
- We have a set $\mathcal{T} = \{1, 2, \dots T\}$ of investment periods
- Let $\omega_{it}, i \in N, t \in \mathcal{T}$ be the return of investment $i \in N$ in period $t \in \mathcal{T}$.
- If we exceed our goal G, we get an interest rate of q that Helen and I can enjoy in our golden years
- If we don't meet the goal of G, Helen and I will have to borrow money at a rate of r so that Jacob can go to MIT.
- We have $\$b$ now.

Variables

- $x_{it}, i \in N, t \in \mathcal{T}$: Amount of money to invest in vehicle i during period t
- y : Excess money at the end of horizon
- w : Shortage in money at the end of the horizon

(Deterministic) Formulation

maximize

$$qy + rw$$

subject to

$$\begin{aligned}
\sum_{i \in N} x_{i1} &= b \\
\sum_{i \in N} \omega_{it} x_{i,t-1} &= \sum_{i \in N} x_{it} \qquad \forall t \in \mathcal{T} \setminus 1 \\
\sum_{i \in N} \omega_{iT} x_{iT} - y + w &= G \\
x_{it} &\geq 0 \qquad \forall i \in N, t \in \mathcal{T} \\
y, w &\geq 0
\end{aligned}$$

One Way to Model

- One way to model this is to create copies of the variables for every scenario at every time period.
- Then we need to enforce *nonanticipativity*...
- Define S_s^t as the set of scenarios that are equivalent (or indistinguishable) to scenario s at time t

A Stochastic Version – Explicit Nonanticipativity

maximize

$$\sum_{s \in S} p_s (q y_s - r w_s)$$

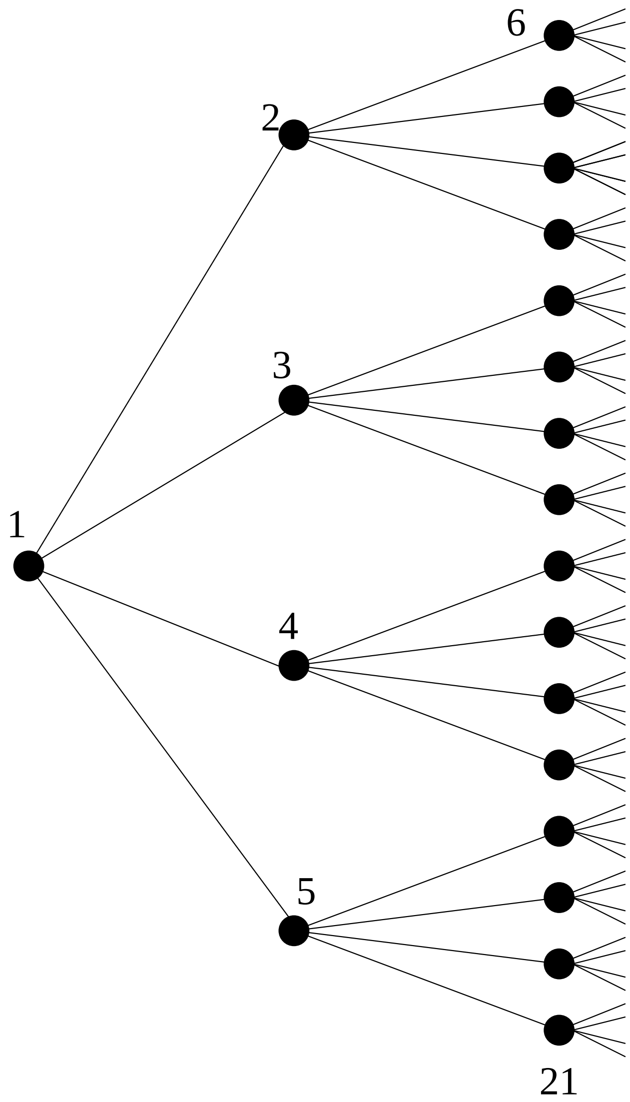
subject to

$$\begin{aligned}
\sum_{i \in N} x_{i1} &= b \\
\sum_{i \in N} \omega_{its} x_{i,t-1,s} &= \sum_{i \in N} x_{its} \qquad \forall t \in \mathcal{T} \setminus 1, \forall s \in S \\
\sum_{i \in N} \omega_{iT} x_{iTs} - y_s + w_s &= G \qquad \forall s \in S \\
x_{its} &= x_{its'} \ \forall i \in N, \forall t \in T, \forall s \in S, \forall s' \in S_s^t \\
x_{its} &\geq 0 \qquad \forall i \in N, t \in \mathcal{T}, \forall s \in S \\
y_s, w_s &\geq 0 \qquad \forall s \in S
\end{aligned}$$

Another Way

- We can also enforce nonanticipativity by just not creating the "wrong" variables
- We have a vector of variables for each node in the tree.
- This vector corresponds to what our decision would be given the realizations of the random variables we have seen so far.
- Index the nodes $l = 1, 2, \ldots \mathcal{L}$.
- We will need to know the "parent" of any node.
- Let $A(l)$ be the ancestor of node $l \in \mathcal{L}$ in the scenario tree.

Jacob-MIT Event Tree



Another Multistage formulation

maximize

$$\sum_{s \in S} p_s (q y_s - r w_s)$$

subject to

$$\begin{aligned}
\sum_{i \in N} x_{i1} &= b \\
\sum_{i \in N} \omega_{il} x_{i,A(l)} &= \sum_{i \in N} x_{il} \qquad \forall l \in \mathcal{L} \setminus 1 \\
\sum_{i \in N} \omega_{iA(s)} x_{iA(s)} - y_s + w_s &= G \qquad \forall s \in S \\
x_{il} &\geq 0 \qquad \forall i \in N, \forall l \in \mathcal{L} \\
y_s, w_s &\geq 0 \qquad \forall s \in S
\end{aligned}$$

Multistage Stochastic LP—Implicit Nonanticipativity

$$\min_{x_1} \left\{ c_1 x_1 + \mathbb{E}_{\omega_2} \left[c_2 x_2 + \mathbb{E}_{\omega_3 | \omega_2} \left[\min_{x_3} + \cdots + \mathbb{E}_{\omega_T | \omega_2, \ldots \omega_{T-1}} \left[\min_{x_T} c_T x_T \right] \right] \right] \right\}$$

$$\begin{array}{llllllll}
A_1 x_1 & & & & & & = & h_1 \\
A_2(\omega_2) x_1 & +W_2 x_2(\omega_2) & & & & & = & h_2(\omega_2) \\
 & A(\omega_3) x_2(\omega_2) & +W_3 x_3(\omega_3) & & & & = & h_3(\omega_3) \\
 & & \ddots & & & & \vdots & \\
 & & & & A(\omega_T) x_{T-1}(\omega_{T-1}) & +W_T x_T(\omega_T) & = & h_T(\omega_T) \\
x_1 & & & & & & \geq & 0 \\
 & x_2(\omega_2) & & & & & \geq & 0 \\
 & & \ddots & & & & \geq & 0 \\
 & & & & & x_T(\omega_T) & \geq & 0
\end{array}$$

Nested Decomposition Procedure

- A (The?) method for solving multistage stochastic programs
- Just like a recursive version of the L-Shaped Method
- Parent nodes send "proposals" for solutions to their children nodes
- Child nodes send cuts to their parents
- There are different "sequence procedures" that tell in which order the problems corresponding to different nodes in the scenario tree are solved.

Nested Decomposition

- "Fast Forward-Fast Back"
  - Most common sequencing procedure
  - Do all nodes at a time period
  - If you find a problem infeasible, create the feasibility cut and reverse direction
  - If problem has (relatively) complete recourse, it amounts to breadth-first search of on the scenario tree

Nested Decomposition

- Cuts have the form $(\theta \geq e - Ex)$, where

$$e = \sum_{k \in \mathcal{D}} (p_k | p_j) \left[\pi_k^T h_k + \sigma_k^T e_k \right]$$

$$E = \sum_{k \in \mathcal{D}} (p_k | p_j) \pi_k^T T_k$$

Hot! Hot! Hot! From B&L Section 7.1

- Planning Production of Air Conditioners (ACs) for next three months
- In each month we can produce at most 200 ACs for \$100 each
- We can use overtime workers if the demand is heavy, but then the cost per unit is \$300
- The demand in the first month is 100
- The demand in the second and third months is random. With 50% probability it will be either 100 or 300.
- We can store ACs at a cost of \$50/month
- We *must* meet all demand
- No salvage/disposal costs at the end of the horizon

Hot! Hot! Hot! Model

Variables

- x_t—Number of ACs to produce (regular) in time t
- w_t—Number of ACs to produce (overtime) in time t
- y_t—Number of ACs to carry over in inventory at the end of time t
- d_t—Demand for ACs in time t

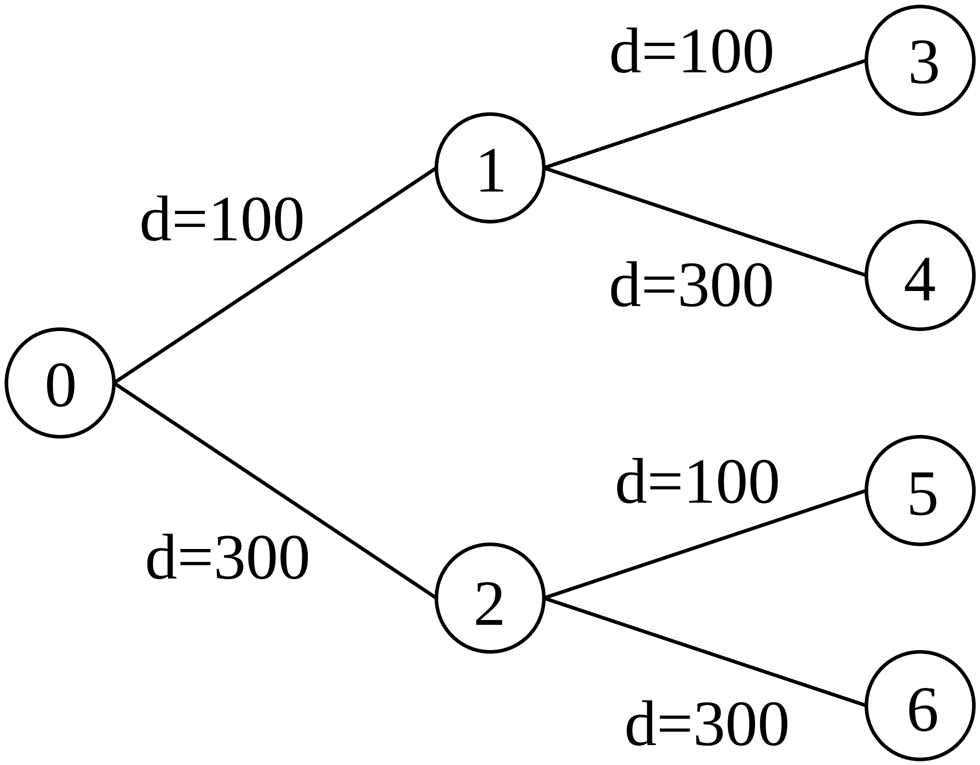
Model at node l

$$\min x_t + 3w_t + 0.5y_t + \theta_l$$

subject to

$$\begin{aligned} x_t &\leq 2 \\ y_{t-1} + x_t + w_t - y_t &= d_l \\ \theta_l &\geq e_k - E_k(x,w,y)^T \qquad \forall \text{cuts at this node} \\ \text{All vars} &\geq 0 \end{aligned}$$

Scenario Tree



- We're going to do a couple iterations of Nested Decomposition.
- **Pay Attention**, it will probably be on the final

April 23 Presentations...

(Only going to do five people, since you will have to fill out evaluations on that day too...)

- Banu Gemici
- Rui Kang
- Jen Rogers
- Jerry Shen
- Clara Novoa

- Next time—Probabilistic Constraints