IE 495 – Lecture 22

Chance Constrained Programming

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Outline

- HW Fixes
- Chance Constrained Programming
  - ◇ Is Hard
  - ◇ Main Result(s)
  - ◇ How to use it

Homework Fixes

- "Graph" problems
  - Please graph the expected value function for *all* (non-integer too) values of x.
- Problem 6.
  - $c = 1$
  - Please graph $x + \mathcal{Q}(x)$ — it will be more instructive.

Multistage Stochastic Programming

- How would you solve a really small instance?
- What is your favorite eight-syllable word?
- Modeling
  - Explicit N-A
  - Implicit N-A
- Nested Decomposition
  - What goes forward?
  - What goes backwards?

A Random Linear Program

minimize

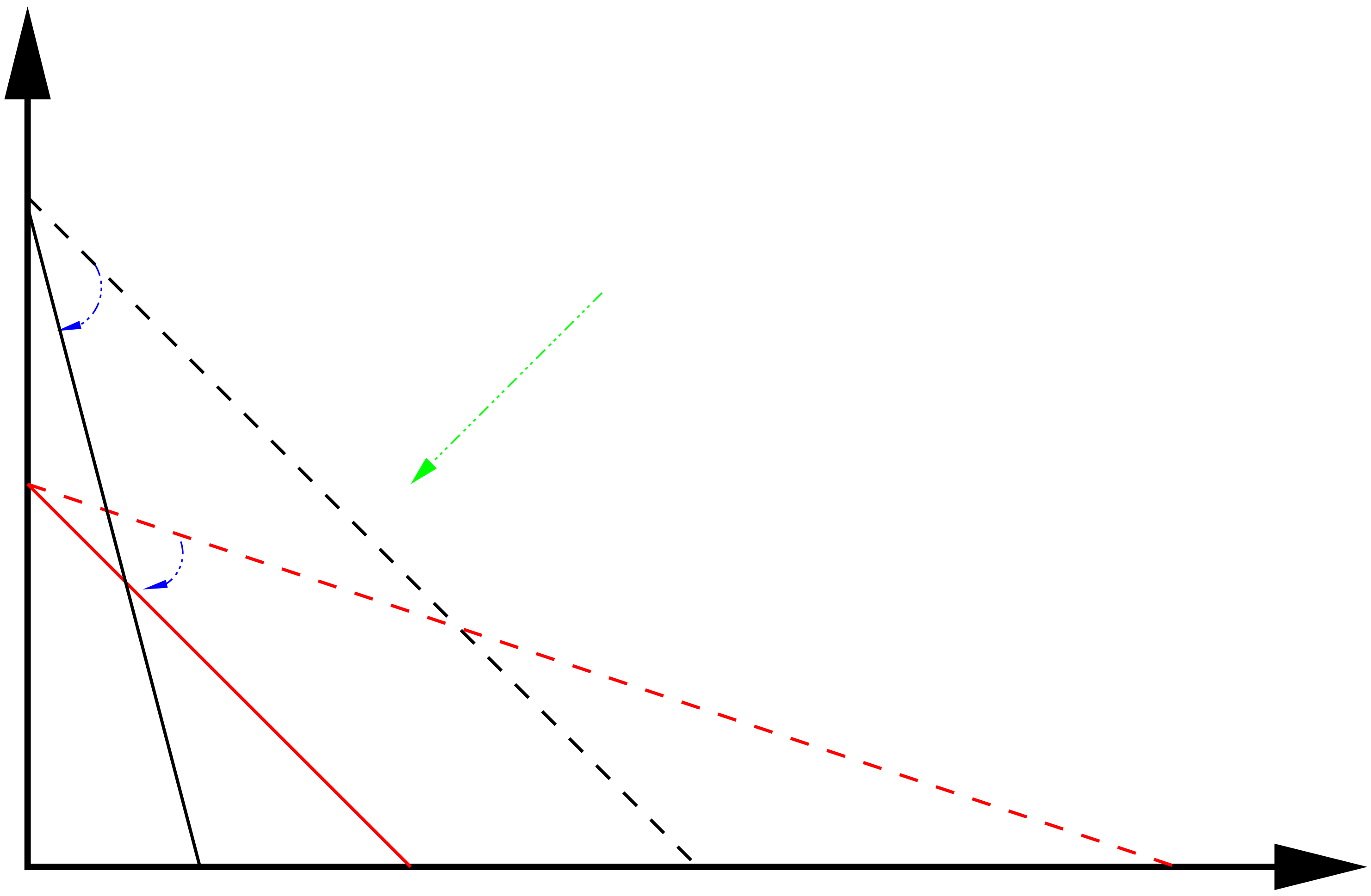
$$x_1 + x_2$$

subject to

$$\begin{aligned} \omega_1 x_1 + x_2 &\geq 7 \\ \omega_2 x_1 + x_2 &\geq 4 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

- $\omega_1 \sim \mathcal{U}[1, 4]$
- $\omega_2 \sim \mathcal{U}[1/3, 1]$

Say Goodbye to Our Favorite Problem...



What To Do?

- How do we solve this problem?
- What do you *mean* by solving this problem?
- Today, we answer the question in another way
- Let's enforce that the *probability* of a constraint holding is sufficiently large.

Chance Constrained

- Separate Chance Constraints

$$\begin{aligned} P\{\omega_1 x_1 + x_2 \geq 7\} &\geq \alpha_1 \\ P\{\omega_2 x_1 + x_2 \geq 4\} &\geq \alpha_2 \end{aligned}$$

- Joint (integrated) chance constraint

$$P\{\omega_1 x_1 + x_2 \geq 7, \omega_2 x_1 + x_2 \geq 4\} \geq \alpha$$

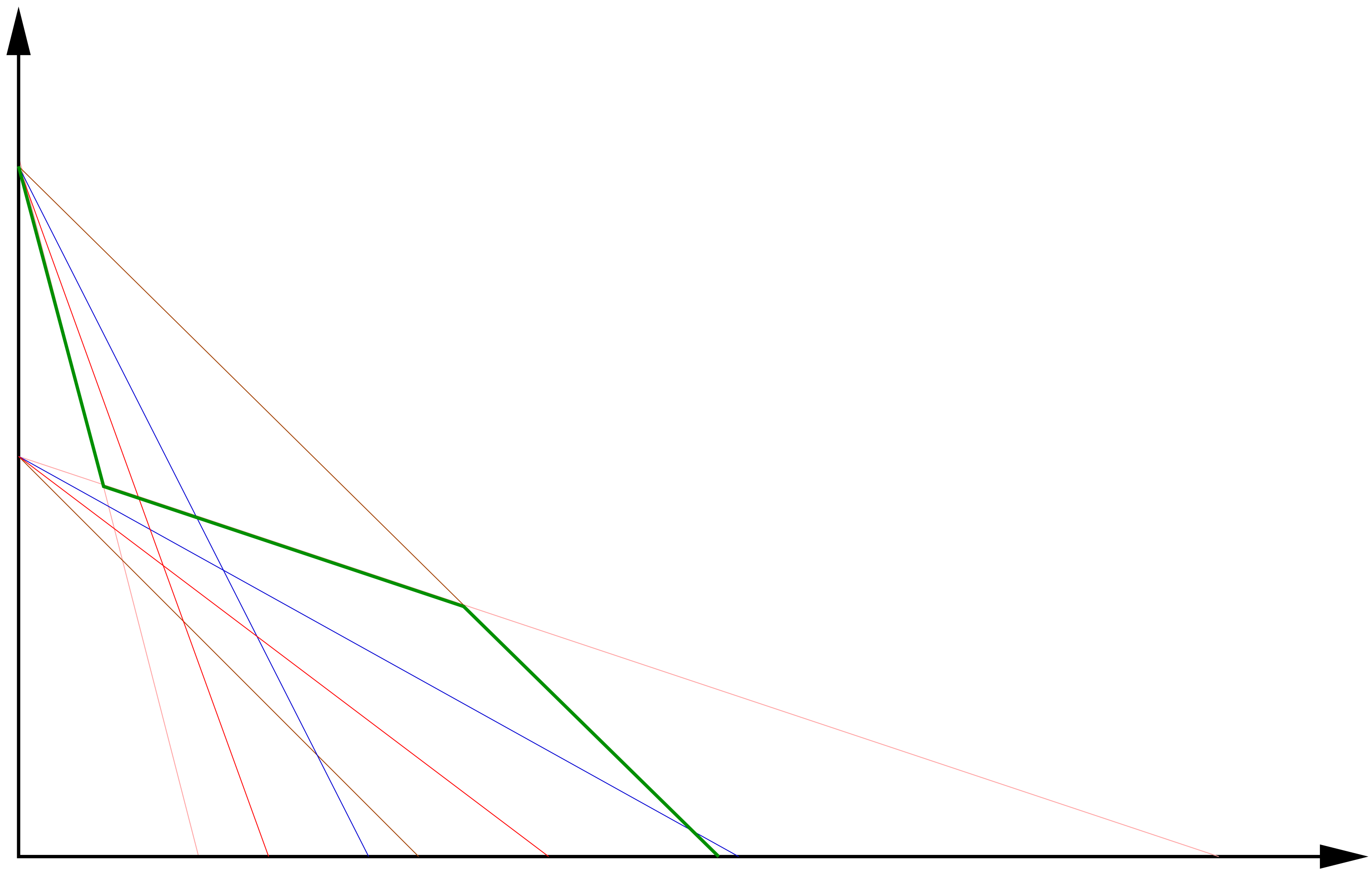
Example—Integrated Chance Constraints

- (1) : $\mathrm{P}\{(\omega_1, \omega_2) = (1, 1)\} = 0.1$
- (2) : $\mathrm{P}\{(\omega_1, \omega_2) = (2, 5/9)\} = 0.4$
- (3) : $\mathrm{P}\{(\omega_1, \omega_2) = (3, 7/9)\} = 0.4$
- (4) : $\mathrm{P}\{(\omega_1, \omega_2) = (4, 1/3)\} = 0.1$
- Consider for some $\alpha \in (0.8, 0.9]$...

$$\mathrm{P}\{\omega_1 x_1 + x_2 \geq 7, \omega_2 x_1 + x_2 \geq 4\} \geq \alpha$$

- Then constraints for realizations (2), (3), and *either* (1) or (4) must hold.

Picture



What Does This Show

- Define the feasibility set...

$$K_1(\alpha) = \{x | \mathrm{P}(T(\omega)x \geq h(\omega)) \geq \alpha\}.$$

- $K_1(\alpha)$ need not be convex. :-(
- When will it be "nice"?

Thm:

Suppose $T(\omega) = T$ is fixed, and $h(\omega)$ has a quasi-concave probability measure P. Then $K_1(\alpha)$ is convex for $0 \leq \alpha \leq 1$

Quasi-Concave

- A function $P : D \rightarrow \Re$ defined on a domain D is quasi-concave if $\forall$ convex $U, V \subseteq D$, and $0 \leq \lambda \leq 1$,

$$\mathrm{P}((1-\lambda)U + \lambda V) \geq \min\{\mathrm{P}(U), \mathrm{P}(V)\}.$$

Quasi-Concave Probability Distributions

Uniform:

$$f(x) = \begin{cases} 1/\mu(S) & x \in S \\ 0 & \text{Otherwise} \end{cases}$$

Exponential density: $f(x) = \lambda e^{-\lambda x}$

Mutivariable normal density: $f(x) = \gamma e^{-\frac{1}{2}(x-\mu)^T V^{-1}(x-\mu)}$

- What this means?
- If you have such a density, you can
  - ◇ Use Lagrangian Techniques
  - ◇ Use a reduced-gradient technique (see Kall & Wallace Section 4.1)

Single Constraint—Easy Case

- The situation in the *single* constraint case is somewhat more simple.
- Suppose again that $T_i(\omega) = T_i$ is constant. (The ith row of the technology matrix is constant, and we wish to enforce...)

$$\mathrm{P}(T_i x \geq h_i(\omega)) = F(T_i x) \geq \alpha$$

so the *deterministic equivalent* is

$$T_i x \geq F^{-1}(\alpha)$$

- This is a linear constraint—we just need to compute $F^{-1}(\alpha)$

Other "Solvable" Cases

- Let $h(\omega) = h$ be fixed, $T(\omega) = (\omega_1, \omega_2, \ldots, \omega_n)$, with $\omega \equiv (\omega_1, \omega_2, \ldots, \omega_n)$ a multivariate normal distribution with mean $\mu = (\mu_1, \mu_2, \ldots \mu_n)$ and covariance matrix V. Then

$$K_1(\alpha) = \{x | \mu^T x \geq h + \Phi^{-1}(\alpha)\sqrt{x^T V x}\}.$$

- $K_1(\alpha)$ is a convex set for $\alpha \geq 0.5$
- You can write it as a "Second Order Cone" constraint.

“Robust” Portfolio Optimization

- Suppose I have a universe of stocks to pick from each with an expected return $\alpha_1, \alpha_2, \ldots \alpha_n$.
- The α are random variables. Assume that they are follow a multivariate normal distribution with means α_i and covariance matrix Σ.
- Suppose I want to be “reasonably” sure that I make a good return T.

Robust Portfolio Optimization

- $x_i \geq 0$: Percentage of portfolio to invest in stock i

$$\mathrm{P}\left\{ (\sum_{i=1}^{n} \alpha_i x_i \geq T) \geq \alpha \right\}$$

$$\alpha^T x - \Phi^{-1}(\alpha)\sqrt{x^T \Sigma x} \geq h$$

- We'll do one final "proof" of the main result if time allows...

Chance Constrained Programming in a Nutshell

- Single Chance Constraint(s)
  - ◇ T_i fixed $\Rightarrow$ LP! $(T_i x \geq F^{-1}(\alpha))$
  - ◇ T_i normal $\Rightarrow$ convex! (Solve as SOCP).
- Joint Chance Constraints
  - ◇ $T(\omega)$ fixed, $h \approx P$, with P quasi-concave $\Rightarrow K_1(\alpha)$ is convex
    - Use Lagrangian Approach
    - a "Reduced Gradient" NLP approach
- Otherwise—Very Hard.
  - ◇ Use a bounding approximation

Next Time...

- I'll pass out the final...
- You should try to have the homework finished...
- For your learning pleasure...
- Banu Gemici
  - Variance Reduction for Monte-Carlo Approaches to Product Portfolio Optimization
- Rui Kang
  - Stochastic Integer Programming—Primal and Dual Approaches

For Your Learning Pleasure...

- Jen Rogers
  - ◇ Replacement Analysis with Technological Breakthroughs
- Jerry Shen
  - ◇ A Multistage Stochastic Programming Approach to Running Red Lights
- Clara Novoa
  - ◇ Integer L-Shaped Method