IE 495 – Lecture 6

Stochastic Programming – MultiPeriod Models

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Outline

- Homework – questions?
  - (I haven't started it yet either)...
  - I won't be in my office after class today...
- Modeling Examples
  - Jacob and MIT
    - A multi-period planning problem
  - "Multi-period" production planning
    - *NOT* a multi-period stochastic program

Please don't call on me!

- Give an example of a function that is not differentiable.
- What is the "subgradient inequality"?
- What are the KKT conditions (in words)?
  - $\hat{x}$ is an optimal solution iff...
  - Why would we care about them?

Here they are again....

Thm: For a convex function $f : \Re^n \mapsto \Re$, and convex functions $g_i : Re^n \mapsto \Re, i = 1, 2, \dots m$, if we have some "regularity conditions", $\hat{x}$ is an optimal solution to

$$\min\{f(x) : g_i(x) \leq 0 \ \forall i = 1, 2, \dots m\}$$

if and only if the following conditions hold:

- $g_i(\hat{x}) \leq 0 \ \forall i = 1, 2, \dots m$
- $\exists \lambda_1, \lambda_2, \dots \lambda_m \in \Re$ such that
  - $0 \in \partial f(\hat{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\hat{x})$.
  - $\lambda_i \geq 0 \ \forall i = 1, 2, \dots m$
  - $\lambda_i g_i(\hat{x}) = 0 \ \forall i = 1, 2, \dots m$

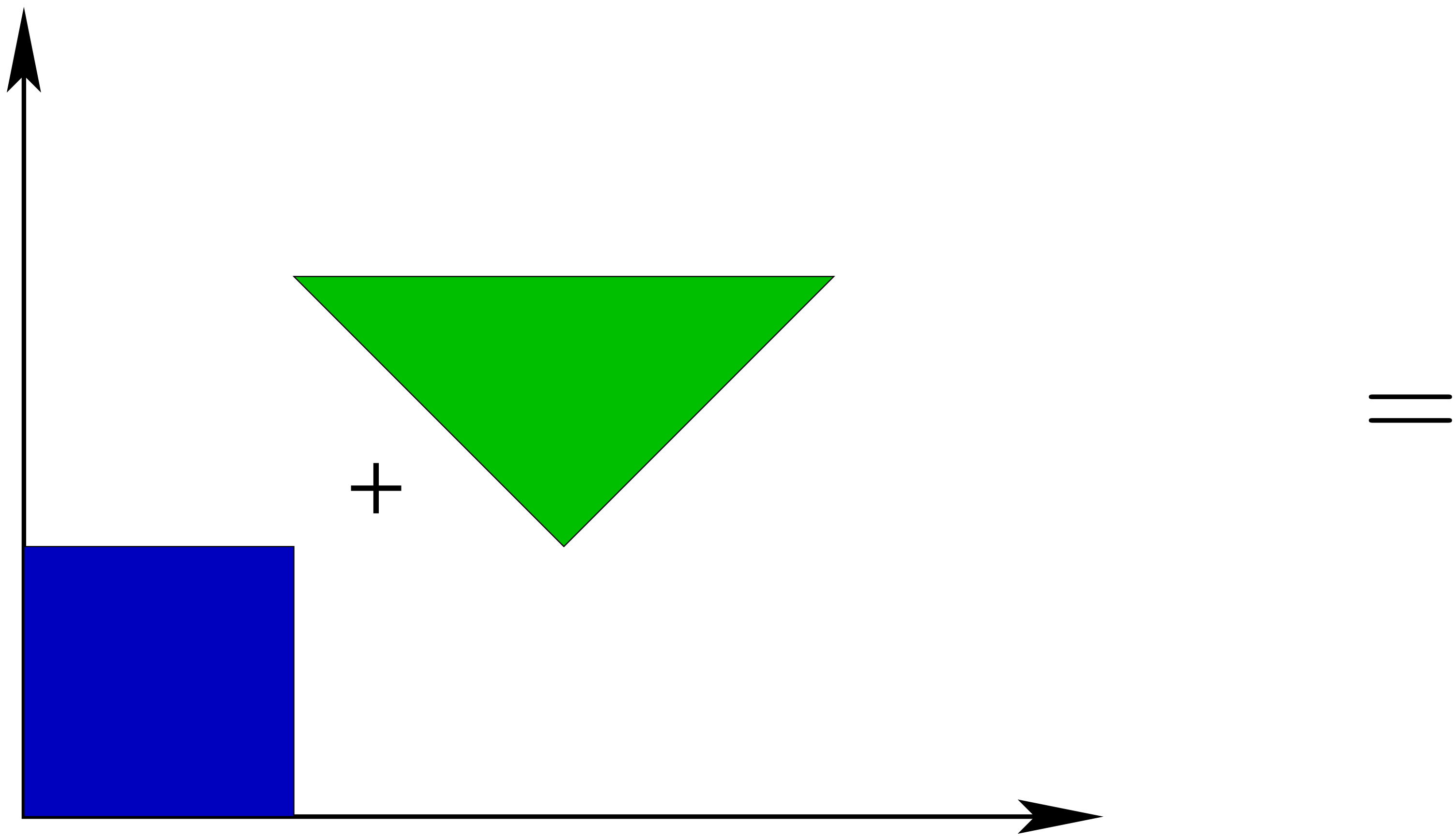
To Be Complete...

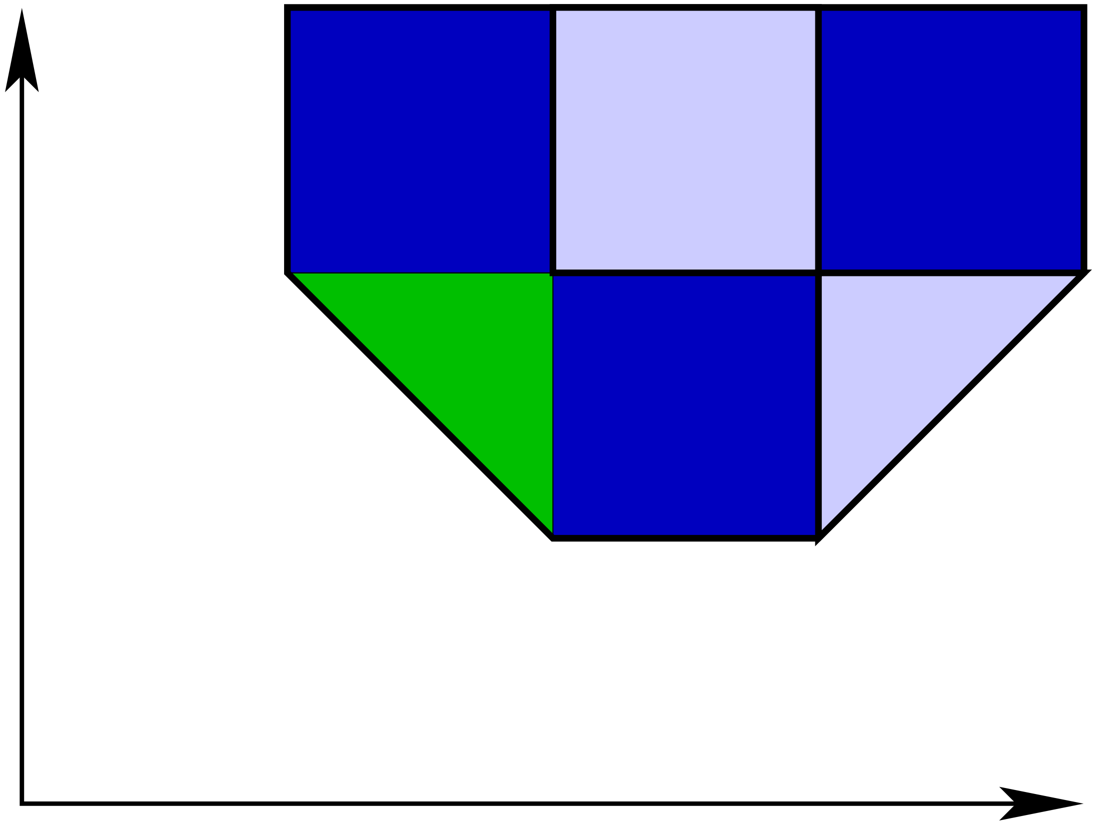
- What the heck does $\sum_{i=1}^{m} \lambda_i \partial g_i(\hat{x})$ mean when the subdifferential is more than a point? (∇)

? What does it mean to add sets?

- Given two sets C_1 and C_2,
  - ◇ $C_1 + C_2 = \{x_1 + x_2 | x_1 \in C_1, x_2 \in C_2\}$.
- This is sometimes called the *Minkowski Sum*
- Geometrically, you “slide one set around the other”.

Example





Daddy Has Big Plans!

- MIT costs \$39,060/year right now
- In 10 years, when Jacob is ready for MIT, it will cost > \$80000/year. (YIKES!)
- Let's design a stochastic programming problem to help us out.
- In Y years, we would like to reach a tuition goal of G.
- We will assume that Helen and I rebalance our portfolio every v years, so that there are $T = Y/v$ times when we need to make a decision about what to buy.
  - ◇ There are T periods in our stochastic programming problem.

Details

- We are given a universe N of investment decisions
- We have a set $\mathcal{T} = \{1, 2, \dots T\}$ of investment periods
- Let $\omega_{it}, i \in N, t \in \mathcal{T}$ be the return of investment $i \in N$ in period $t \in \mathcal{T}$.
- If we exceed our goal G, we get an interest rate of q that Helen and I can enjoy in our golden years
- If we don't meet the goal of G, Helen and I will have to borrow money at a rate of r so that Jacob can go to MIT.
- We have \$$b$ now.

Variables

- $x_{it}, i \in N, t \in \mathcal{T}$: Amount of money to invest in vehicle i during period t
- y : Excess money at the end of horizon
- w : Shortage in money at the end of the horizon

(Deterministic) Formulation

maximize

$$qy + rw$$

subject to

$$\begin{aligned}
\sum_{i \in N} x_{i1} &= b \\
\sum_{i \in N} \omega_{it} x_{i,t-1} &= \sum_{i \in N} x_{it} \qquad \forall t \in \mathcal{T} \setminus 1 \\
\sum_{i \in N} \omega_{iT} x_{iT} - y + w &= G \\
x_{it} &\geq 0 \qquad \forall i \in N, t \in \mathcal{T} \\
y, w &\geq 0
\end{aligned}$$

Random returns

- As evidenced by our recent performance, my wife and I are bad at picking stocks.
  - ◇ In our defense, returns on investments are *random* variables.
- Imagine that for each there are a number of potential outcomes R for the returns at each time t.

⋆ We now can (and potentially would) change our portfolio after observing the (now random) returns ω_{it}

Pontificating

- An important point about this problem is that we want to take corrective action *TO REACH THE GOAL* of G dollars.
- If we were trying to maximize return, there would be no reason to change from the current strategy at later periods.

  ⇒ There is no reason to use (multistage) stochastic programming
- In general, unless you wish to consider the impact of changing your decision (or correcting for your decision) at later periods *on your decision at the current period (now)*, you should *not* use stochastic programming.

A Scenario Tree

- We will assume (or approximate) the random returns with discrete values, each with an associated probability.
- Conceptually, the sequence of random events (returns) can be arranged into a tree

(Your picture here)

Scenarios

- The scenarios consist of all possible sequences of outcomes.

Ex. Imagine $R = 4$ and $T = 3$. The the scenarios would be...

$t = 1$	$t = 2$	$t = 3$
1	1	1
1	1	2
1	1	3
1	1	4
1	2	1
	$\vdots$	
4	4	4

- $|\Omega| = 64$
- We can specify any probability on this outcome space that we would like
  - For example, things time period outcomes don't need to be equally likely, or returns in different time periods need not be mutually independent.

Making it Stochastic

- $x_{its}, i \in N, t \in \mathcal{T}, s \in S$: Amount of money to invest in vehicle i during period t in scenario s
- y_s : Excess money at the end of horizon in scenario s
- w_s : Shortage in money at the end of the horizon in scenario s

⋆ Note that the (random) return ω_{it} now is like a function of the scenario s.

  ◇ It depends on the mapping of the scenarios to the scenario tree.

A Stochastic Version

maximize

$$qy_s + rw_s$$

subject to

$$\begin{aligned}
\sum_{i \in N} x_{i1} &= b \\
\sum_{i \in N} \omega_{its} x_{i,t-1,s} &= \sum_{i \in N} x_{its} \qquad \forall t \in \mathcal{T} \setminus 1, \forall s \in S \\
\sum_{i \in N} \omega_{iT} x_{iTs} - y_s + w_s &= G \qquad \forall s \in S \\
x_{its} &\geq 0 \qquad \forall i \in N, t \in \mathcal{T}, \forall s \in S \\
y_s, w_s &\geq 0 \qquad \forall s \in S
\end{aligned}$$

Easy, Huh?

- See how easy it is to convert a deterministic model to a (multistage) stochastic model
- Just one thing wrong
  - The model is incorrect.
- Imagine the following...

(Your picture here)

My Favorite Eight Syllable Word

- We need to enforce *nonanticipativity*.
  - ◇ Other eight-syllable words...
  - ◇ autosuggestibility, incommensurability, electroencephalogram, unidirectionality
- At any point in time, different scenarios "look the same"
  - ◇ We can't allow different decisions for these scenarios.
  - ◇ We are not allowed to anticipate the outcome of future random events when making our decision now.
- Let S_s^t be the set set of scenarios that are identical to scenario s at time t.

⋆ We must enforce $x_{its} = x_{its'} \ \forall i \in N, \forall t \in T, \forall s \in S, \forall s' \in S_s^t$.

Another way

- There's more than one way to enforce that our policy is nonanticipative.
- Birge's way:

$$\left(\sum_{s' \in S_s^t} p'_s\right) x_{its} = \sum_{s' \in S_s^t} p'_s x_{its'} \qquad \forall i, t, s$$

- The probability you reach a scenario equivalent to s times the amount you invest in scenario s must be equal to the expected amount you would invest in all scenarios equivalent to s
- Which way is best?

Multiperiod Production Planning

- A factory makes several different products
- Resources (e.g. machines and labor) are needed
  - ◇ For each product, these requirements are "known"
- Must meet demand at the end of each period.
  - ⋆ Demand is not known with certainty
- Costs are induced when inventory is too large or too small
  - ◇ (Inventory and purchase costs)
- To satisfy demand, additional labor and machine hours can be used, but these additions are bounded.

⋆ There is a "hire and fire" cost associated with changing the workforce level.

Decision Problem

- Right now, we will decide
  - ◇ The number of each product to be produced in *each* period
  - ◇ The extra capacity to be used in *each* period
  - ◇ The hirings and firings to be done in *each* period
- Then a random demand occurs
  - ◇ Conceptually, this occurs for all periods
- Then after we observe the random demands, we can decide how to best store product in inventory or purchase from an outside source.

? In this framework, how many stages are in the stochastic programming instance?

Lots of Definitions

Sets

- T : Number of periods. (Also set T).
- N : Set of products
- M : Set of resources

Variables

- x_{jt} : Product of product $j \in N$ in period $t \in T$
- u_{it} : Additional amount of resource $i \in M$ to purchase in $t \in T$
- $z^+_{t-1,t}, z^-_{t-1,t}$: Planned increase/decrease of work force
- y^-_{jt}, y^+_{jt}: Surplus/Shortage of project $j \in N$ at the end of period $t \in T$.

Parameters

- All of the above variables have associated costs $(\alpha, \beta, \gamma, \delta)$.
- ω_{jt} : Demand for product $j \in N$ in period $t \in T$.
- U_{it} : Upper bound on u_{it}.
- a_{ij} : Amount of resource $i \in M$ needed to produce one unit of product $j \in N$
- b_{it} : Amount of resource $i \in M$ available at time $t \in T$.

The Model

minimize

$$\sum_{j \in N} \sum_{t \in T} \alpha_{jt} x_{jt} + \sum_{i \in M} \sum_{t \in T} \beta_{it} u_{it} + \sum_{t \in T \setminus 1} (\gamma^{+}_{t-1,t} z^{+}_{t-1,t} + \gamma^{-}_{t-1,t} z^{-}_{t-1,t}) +$$

$$\mathbb{E}_{\omega} \left[\min \left(\sum_{j \in N} \sum_{t \in T} (\delta^{+}_{jt} y^{+}_{jt} + \delta^{-}_{jt} y^{-}_{jt}) \right) \right]$$

subject to

$$\begin{aligned}
\sum_{j \in N} a_{ij} x_{jt} &\leq b_{it} + u_{it} && \forall i \in M, \forall t \in T \\
u_{it} &\leq U_{it} && \forall i \in M, \forall t \in T \\
z^{+}_{t-1,t} - z^{-}_{t-1,t} &= \sum_{j \in N} a_{Lj} (x_{jt} - x_{j,t-1}) && \forall t \in T \setminus 1 \\
x_{jt} + y^{-}_{j,t-1} + y^{+}_{jt} - y^{-}_{jt} &= \omega_{jt} && \forall j \in N, \forall t \in T.
\end{aligned}$$

The Stochastic LP/Recourse Model

minimize

$$\sum_{j\in N}\sum_{t\in T}\alpha_{jt}x_{jt}+\sum_{i\in M}\sum_{t\in T}\beta_{it}u_{it}+\sum_{t\in T\setminus 1}(\gamma^{+}_{t-1,t}z^{+}_{t-1,t}+\gamma^{-}_{t-1,t}z^{-}_{t-1,t})+$$

$$\sum_{s\in S}\sum_{j\in N}\sum_{t\in T}p_s(\delta^{+}_{jt}y^{+}_{jts}+\delta^{-}_{jt}y^{-}_{jts})$$

subject to

$$\begin{aligned}
\sum_{j\in N}a_{ij}x_{jt} &\leq b_{it}+u_{it} && \forall i\in M,\forall t\in T\\
u_{it} &\leq U_{it} && \forall i\in M,\forall t\in T\\
z^{+}_{t-1,t}-z^{-}_{t-1,t} &= \sum_{j\in N}a_{Lj}(x_{jt}-x_{j,t-1}) && \forall t\in T\setminus 1\\
x_{jt}+y^{-}_{j,t-1,s}+y^{+}_{jts}-y^{-}_{jts} &= \omega_{jts} && \forall j\in N,\forall t\in T,\forall s\in S.
\end{aligned}$$

Next time?

- Maybe a little AMPL modeling and show and tell of multistage problems
- Properties of the resource function
- Tools for modeling and solving (real) (non-toy) stochastic programs