On Valid Inequalities for Quadratic Programming with Continuous Variables and Binary Indicators

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Abstract. In this paper we study valid inequalities for a set that involves a continuous vector variable $x \in [0,1]^n$, its associated quadratic form xx^T, and binary indicators on whether or not $x > 0$. This structure appears when deriving strong relaxations for mixed integer quadratic programs (MIQPs). Valid inequalities for this set can be obtained by lifting inequalities for a related set without binary variables (**QPB**), that was studied by Burer and Letchford. After closing a theoretical gap about **QPB**, we characterize the strength of different classes of lifted **QPB** inequalities. We show that one class, *lifted-posdiag-QPB inequalities*, capture no new information from the binary indicators. However, we demonstrate the importance of the other class, called *lifted-concave-QPB inequalities*, in two ways. First, all lifted- concave-QPB inequalities define the relevant convex hull for the case of *convex* quadratic programming with indicators. Second, we show that all *perspective constraints* are a special case of lifted-concave-QPB inequalities, and we further show that adding the perspective constraints to a semidefinite programming (SDP) relaxation of convex quadratic programs with binary indicators results in a problem whose bound is equivalent to the recent optimal diagonal splitting approach of Zheng *et al.*. Finally, we show the separation problem for lifted-concave-QPB inequalities is tractable if the number of binary variables involved in the inequality is small. Our study points out a direction to generalize perspective cuts to deal with non-separable nonconvex quadratic functions with indicators in global optimization. Several interesting questions arise from our results, which we detail in our concluding section.

Keywords: Mixed integer quadratic programs, Semidefinite Programming, Valid inequalities, Perspective Reformulation

1 Introduction

Our primary goal in this work is to solve Mixed Integer Quadratic Programming (MIQP) problems with indicator variables of the form

$$\min_{x\in\mathbb{R}^n, z\in\{0,1\}^n} \{c^T x + d^T z + x^T Q x \mid Ax + Bz \le b,\ 0 \le x_i \le u_i z_i\ \forall i = 1, \ldots n\}. \quad (1)$$

In (1), the binary variable z_i is used to indicate the positivity of its associated continuous variable $x_i, \forall i = 1, \dots, n$. Related problems of this type arise in many applications, including portfolio selection [5], sparse least-squares [22], optimal control [20], and unit-commitment for power generation [15]. The optimization problem (1) can be very difficult to solve to optimality. Computational experience presented in [4] shows that for problems of size $n = 100$, a branch-and-bound algorithm typically requires more than 10^6 nodes to solve the problem to optimality.

A standard technique for solving (1) is to linearize the objective by introducing a new variable for each product of variables $x_i x_j$, arranging these new variables into a matrix variable X. Problem (1) can then be written as

$$\min_{(x,z,X)\in T} \{c^T x + d^T z + Q \bullet X\}, \tag{2}$$

where

$$T := \left\{ (x,z,X) \in \mathbb{R}^{2n+\frac{n(n+1)}{2}} \;\middle|\; \begin{array}{l} z \in \{0,1\}^n,\ X = xx^T,\ Ax + Bz \le b \\ 0 \le x_i \le u_i z_i,\ i = 1, ..., n \end{array} \right\}.$$

All matrices considered in this paper are symmetric, so they can be represented as a vector in a linear space of dimension $\frac{n(n+1)}{2}$ by stacking columns of upper triangular part of the matrix. Given two $n \times n$ symmetric matrices X and Y, their inner product is defined as $X \bullet Y = \sum_{i=1}^n X_{ii} Y_{ii} + 2 \sum_{i<j} X_{ij} Y_{ij}$.

To solve Problem 2, it suffices to optimize the objective over $\mathbf{conv}(T)$, so it is natural to study T and closely-related sets. In this paper, we primarily study valid inequalities for the following set and its convex hull:

$$S := \left\{ (x,z,X) \in \mathbb{R}^{2n+\frac{n(n+1)}{2}}, \begin{array}{l} x \in [0,1]^n, z \in \{0,1\}^n, \\ X = xx^T, x_i \le z_i, i = 1, ..., n \end{array} \right\}.$$

In S, the general bounds on the continuous variables in T have changed to $x \in [0,1]^n$. This change results in no loss of generality. However, the set S does not have the linear constraints $Ax + Bz \le b$ in the definition of T.

By moving the nonlinearity in (1) into the constraints, many of the results we obtain can be directly applied to create strong convex relaxations of problems that additionally have quadratic constraints and indicator variables. These problem arise in applications such as product pooling with network design [13, 24] and digital filter design [27].

When the quadratic functions are convex, a more natural relaxation to study is the following "larger" set,

$$S^{\succeq} := \left\{ (x,z,X) \in \mathbb{R}^{2n+\frac{n(n+1)}{2}}, \begin{array}{l} x \in [0,1]^n, z \in \{0,1\}^n, \\ X \succeq xx^T, x_i \le z_i, i = 1, ..., n \end{array} \right\},$$

where the notation $X \succeq xx^T$ means that the matrix $X - xx^T$ is positive semidefinite.

The remainder of the extended abstract is organized into five sections. Section 2, describes basic properties of the set S. The relationship between S, the

Boolean Quadric Polytope **BQP** [23], and the box-constrained QP set **QPB** [11] is shown, and we slightly strengthen an earlier result known about valid inequalities for **QPB**. We next discuss valid inequalities of S obtained by lifting certain inequalities for **QPB**. The inequalities are divided into two classes, called *lifted-posdiag-QPB* inequalities, and *lifted-concave-QPB* inequalities. Section 3 shows the negative results that *lifted-posdiag-QPB* inequalities contribute essentially no additional strength to the continuous relaxation. In Section 4, we establish the importance of lifted-concave-QPB inequalities for defining strong relaxations of S. We show that the "simplest" class of lifted-concave-QPB inequalities already contains all perspective cuts [14]. As a by-product, for convex quadratic programs with binary indicators, we propose a semidefinite programming (SDP) relaxation that is no worse than the relaxation obtained by *any* diagonal splitting and perspective reformulation scheme [16]. Further, the corresponding dual SDP provides the optimal diagonal splitting. A similar (but slightly weaker) result was previously obtained in [28]. In Section 4, we also show that every valid linear inequality for $\mathbf{conv}(S^{\succeq})$ is a lifted-concave-QPB inequality. Finally, in Section 5, we provide a tractability result on the separation of lifted-concave-QPB inequalities, establishing that the inequalities can be separated in time that is polynomial in the number of binary variables simultaneously lifted. Section 5 also contains an example of size $n = 3$ where the relaxation with lifted-concave-QPB inequalities dominates the doubly-nonnegative relaxation of [9]. We conclude in Section 6 with some natural directions for research that are motivated by this work.

2 Basic Properties

Proposition 1 establishes three fundamental properties of $\mathbf{conv}(S)$ and $\mathbf{conv}(S^{\succeq})$.

Proposition 1

- *Both* $\mathbf{conv}(S)$ *and* $\mathbf{conv}(S^{\succeq})$ *are full-dimensional.*
- *The set of extreme points for* $\mathbf{conv}(S)$ *is* S.
- $\mathbf{conv}(S^{\succeq}) = \mathbf{conv}(S) + \left\{(0,0,X) \in \mathbb{R}^{2n+\frac{n(n+1)}{2}}, X \succeq 0\right\}$.

Proof. The straightforward proof is given in the appendix.

By projecting away z from $\mathbf{conv}(S)$, we obtain the set **QPB** studied in [11],

$$\mathbf{proj}_{(x,X)}\left(\mathbf{conv}(S)\right) = \mathbf{QPB} = \mathbf{conv}\{(x,X) \in \mathbb{R}^{n+\frac{n(n+1)}{2}} : \\ x \in [0,1]^n, X_{ij} = x_i x_j, 1 \le i \le j \le n\}.$$

Furthermore, as proved by [11], projecting away the diagonal entries of X in **QPB** yields the well-known *Boolean Quadric Polytope* (**BQP**) [23]:

$$\mathbf{proj}_{(x,\mathbf{ADiag}(X))}\left(\mathbf{QPB}\right) = \mathbf{BQP} = \mathbf{conv}\{(x,y) \in \mathbb{R}^{n+\frac{n(n-1)}{2}} : \\ x \in \{0,1\}^n, y_{ij} = z_i z_j, 1 \le i < j \le n\},$$

where $\mathbf{ADiag}(X)$ denotes a vector of dimension $n(n-1)/2$ obtained by stacking entries above (but not including) the diagonal of X. These two observations reveal the set $\mathbf{conv}(S)$ to contain interesting interactions between continuous and binary variables in the quadratic context.

Burer and Letchford [11] also classified linear inequalities valid for **QPB** according to the eigenvalues of the matrix of coefficients for X. Specifically, the inequality

$$B \bullet X + \alpha^T x + \gamma \leq 0 \tag{3}$$

is called *convex-QPB*, *concave-QPB*, or *indefinite-QPB*, if its associated quadratic form $x^T Bx + \alpha^T x + \gamma$ is convex, concave or indefinite, respectively. Burer and Letchford proved the following results for convex and concave-QPB inequalities.

Proposition 2 ([11],Proposition 8) *A point $(\bar{x}, \bar{X}) \in \mathbb{R}^{n+\frac{n(n+1)}{2}}$ satisfies all concave-QPB inequalities if and only if it is in the convex set*

$$\left\{(x, X) \middle| X \succeq xx^T, x \in [0,1]^n\right\}.$$

The original proposition in [11] does not demonstrate the "only if" part of Proposition 2, but the result easily follows from the fact that $X \succeq xx^T$ is equivalent to (x, X) satisfying the infinitely-many concave inequalities

$$-\begin{pmatrix} s \\ v \end{pmatrix}^T \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} s \\ v \end{pmatrix} = -(vv^T) \bullet X - 2(sv)^T x - s^2 \leq 0, \forall s \in \mathbb{R}, v \in \mathbb{R}^{n-1}.$$

This observation also establishes that it suffices to consider concave-QPB inequalities with $\mathbf{rank}(B) \leq 1$.

For convex-QPB inequalities, Burer and Letchford provided the following partial characterization.

Proposition 3 ([11], Proposition 9) *If $B \bullet X + \alpha^T x + \gamma \leq 0$ is a valid inequality for* **QPB** *and $B \succeq 0$, then it is valid for the convex set*

$$\left\{(x, X) | (x, \mathbf{ADiag}(X)) \in \mathbf{BQP}, X_{ii} \leq x_i, \forall i = 1, \ldots, n\right\}.$$

Proposition 3 only establishes the necessity for (3) to be a convex-QPB inequality, not its sufficiency. We fill this gap in Proposition 4 by considering a larger class that includes the convex-QPB inequalities.

Proposition 4 *A point $(\bar{x}, \bar{X})$ satisfies all inequalities $B \bullet X + \alpha^T x + \gamma \leq 0$ with $B_{ii} \geq 0, \forall i = 1, ..., n$ valid for* **QPB** *if and only if it is in the convex set*

$$\left\{(x, X) | (x, \mathbf{ADiag}(X)) \in \mathbf{BQP}, X_{ii} \leq x_i, \forall i = 1, \ldots, n\right\}.$$

Proof. The proof is given in the appendix.

We call inequalities (3) with $B_{ii} \geq 0$ valid for **QPB** *posdiag-QPB* inequalities.

Let $\mathcal{Q}$ be the intersection of the two convex sets in Propositions 2 and 3, i.e., $\mathcal{Q}$ is the relaxation of **QPB** defined by all concave and posdiag-QPB inequalities.

Separating concave-QPB inequalities can be done in polynomial time, but separating convex, or posdiag-QPB inequalities is NP-Complete, as **BQP** is affinely equivalent to the cut polytope [23].

Burer and Letchford demonstrate that $\mathbf{QPB} \subsetneq \mathcal{Q}$, even for $n = 3$, although it follows from [2] that $\mathbf{QPB} = \mathcal{Q}$ for $n \leq 2$. On the other hand, $\mathcal{Q}$ empirically has been shown to be a very tight relaxation of **QPB**. Specifically, Anstreicher [1] shows that using a subset of all valid inequalities for $\mathcal{Q}$ suffices to solve 49 of 50 instances (up to size $n = 60$) of the BoxQP library [12] at the root node. The inequalities used in the study of Anstreicher are all concave-QPB inequalities and posdiag-QPB inequalities derived via the Reformulation-Linearization Technique [26] and the triangle inequalities for **BQP** introduced by [23].

In the remainder of the paper, we study valid inequalities for the case $\mathbf{conv}(S)$ (and $\mathbf{conv}(S^{\succeq})$), when the indicator variables z come into play. Note that by setting $z_i = 1\ \forall i$, $\mathbf{conv}(S)$ is easily mapped to **QPB**. Our hope is to capitalize on the strength of $\mathcal{Q}$ as a relaxation of **QPB** to generate strong relaxations for $\mathbf{conv}(S)$. More specifically, for any valid inequality for $\mathbf{conv}(S)$

$$B \bullet X + \alpha^T x + \gamma \leq \delta^T z, \tag{4}$$

the inequality $B \bullet X + \alpha^T x + (\gamma - \delta^T e) \leq 0$ is a valid inequality for **QPB**, where e is a vector of all ones with proper dimension. In this sense, valid inequalities for $\mathbf{conv}(S)$ can be obtained by lifting valid inequality for **QPB**, i.e., by determining δ and modifying the constant term appropriately. We analyze the strength of lifted-concave and lifted- posdiag-QPB inequalities separately in the following two sections.

3 Lifted-Posdiag-QPB Inequalities

In this section we characterize the set defined by all lifted-posdiag- QPB inequalities for $\mathbf{conv}(S)$. The analysis shows the "negative" result that lifted-posdiag-QPB inequalities provide no restriction on z_i other than that provided by the continuous relaxation: $x_i \leq z_i \leq 1$.

Theorem 1. *A point $(\bar{x}, \bar{X}, \bar{z}) \in \mathbb{R}^{2n+\frac{n(n+1)}{2}}$ satisfies all valid inequalities $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ for $\mathbf{conv}(S)$, with $B_{ii} \geq 0, \forall i = 1, \ldots, n$, if and only if it is in the following convex set:*

$$\left\{(x, X, z) | (x, \mathbf{ADiag}(X)) \in \mathbf{BQP}, X_{ii} \leq x_i \leq z_i \leq 1, \forall i = 1, \ldots n\right\}. \tag{5}$$

Proof. We first show that if $(\bar{x}, \bar{X}, \bar{z})$ satisfies all valid inequalities for $\mathbf{conv}(S)$ with $B_{ii} \geq 0$, then the point is in the set defined in (5). Since **BQP** is a projection of **QPB**, any valid inequality for $(x, \mathbf{ADiag}(X)) \in \mathbf{BQP}$ is a lifted-posdiag-QPB inequality for $\mathbf{conv}(S)$, as the coefficients for X_{ii} are zeros. The inequalities $X_{ii} - x_i \leq 0$, $x_i \leq z_i$ and $-1 \leq -z_i$ are also lifted-posdiag-QPB inequalities.

To prove the other direction, let $(\bar{x}, \bar{X}, \bar{z})$ be such that $(\bar{x}, \mathbf{ADiag}(\bar{X})) \in \mathbf{BQP}, \bar{X}_{ii} \leq \bar{x}_i \leq \bar{z}_i \leq 1 \forall i = 1, \ldots, n$. We show this point satisfies all lifted-posdiag-QPB inequalities for $\mathbf{conv}(S)$. The first claim is that it suffice to show this for all lifted-posdiag-QPB inequalities with $\delta_i \geq 0\ \forall i = 1,\ldots,n$. A proof of the claim is given in the appendix.

Claim. $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ is valid for $\mathbf{conv}(S)$ if and only if the inequality

$$B \bullet X + \alpha^T x + \gamma \leq \sum_{i:\delta_i \geq 0} \delta_i z_i + \sum_{i:\delta_i < 0} \delta_i \tag{6}$$

is also valid for $\mathbf{conv}(S)$.

Next notice that for any $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ valid for $\mathbf{conv}(S)$, if $x = z \in \{0,1\}^n$, we have that $x^T B x + (\alpha - \delta)^T x + \gamma \leq 0$ for all $x \in \{0,1\}^n$.

As we assumed $(\bar{x}, \mathbf{ADiag}(\bar{X})) \in \mathbf{BQP}$, there exists a set with at most $K = n + \frac{n(n+1)}{2} + 1$ binary vectors: $\{y_k\}_{k=1}^K$ such that $\bar{x} = \sum_{k=1}^K \lambda_k y_k$ and $\bar{X} - \mathbf{Diag}(\bar{X}) + \mathbf{Diag}(\bar{x}) = \sum_{k=1}^K \lambda_k y_k y_k^T$. Here $\lambda_k \geq 0$, $\sum_k \lambda_k = 1$, $\bar{X} - \mathbf{Diag}(\bar{X}) + \mathbf{diag}(\bar{x})$ means replacing the diagonal of $\bar{X}$ with entries in $\bar{x}$, i.e., $\mathbf{Diag}(\bar{X})$ is a diagonal matrix with the diagonal entries of $\bar{X}$, and $\mathbf{Diag}(\bar{x})$ is a diagonal matrix with entries of vector $\bar{x}$. Then,

$$\begin{aligned}
&B \bullet \bar{X} + \alpha^T \bar{x} + \gamma - \delta^T \bar{z} \leq B \bullet \bar{X} + (\alpha - \delta)^T \bar{x} + \gamma \\
&= B \bullet (\bar{X} - \mathbf{Diag}(\bar{X}) + \mathbf{Diag}(\bar{x})) + (\alpha - \delta)^T \bar{x} + \gamma + \sum_{i=1}^n B_{ii}(\bar{X}_{ii} - \bar{x}_i) \\
&\leq B \bullet \left(\sum_k \lambda_k x^{(k)} x^{(k)T} \right) + (\alpha - \delta)^T \left(\sum_k \lambda_k x^{(k)} \right) + \gamma \\
&= \sum_k \lambda_k \left(B \bullet x^{(k)} x^{(k)T} + (\alpha - \delta)^T x^{(k)} + \gamma \right) \leq 0.
\end{aligned}$$

The first inequality inequaities follows because $\delta_i \geq 0$ and $\bar{x}_i \leq \bar{z}_i$. The second inequality is because $B_{ii} \geq 0$ and $\bar{X}_{ii} \leq \bar{x}_i$. The final inequality follows from the observation in the previous paragraph. This concludes our proof.

A similar negative result about the lifted-posdiag-QPB inequalities holds for $\mathbf{conv}(S^{\succeq})$.

Proposition 5 *An inequality $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ with $B_{ii} \geq 0$, $\forall i = 1,\ldots,n$ is valid for $\mathbf{conv}(S^{\succeq})$ if and only if $B = 0$ and $\alpha^T x + \gamma \leq \delta^T z$ is valid for the convex set $\{(x,z) \mid 0 \leq x \leq z \leq 1\}$.*

Proof. The proof is straightforward using the fact that $\mathbf{conv}(S^{\succeq})$ has a recession cone $\left\{(0,0,X) \in \mathbb{R}^{2n+\frac{n(n+1)}{2}}, X \succeq 0\right\}$. It is given in the appendix.

4 Lifted-Concave-QPB Inequalities

In this section, we consider the lifted-concave-QPB inequalities for $\mathbf{conv}(S)$ and show that the class defines $\mathbf{conv}(S^{\succeq})$.

Proposition 6 *A point* $(\bar{x}, \bar{X}, \bar{z}) \in \mathbb{R}^{2n+\frac{n(n+1)}{2}}$ *satisfies all valid inequalities* $B \bullet X + \alpha^T x + \gamma \le \delta^T z$ *for* $\mathbf{conv}(S)$*, with* $B \preceq 0$ *if and only if* $(\bar{x}, \bar{X}, \bar{z}) \in \mathbf{conv}(S^{\succeq})$.

Proof. The proof uses the fact that

$$\mathbf{conv}(S^{\succeq}) = \mathbf{conv}(S) + \left\{(0,0,X) \in \mathbb{R}^{2n+\frac{n(n+1)}{2}}, X \succeq 0\right\}$$

and is given in the appendix.

Next we consider the special case where each of B, α, and δ have at most one nonzero entry. We show that this class of inequalities includes all perspective cuts that use diagonal entries of X. Further, we show that by adding this simple class of inequalities to the semidefinite programming (SDP) relaxation of (1) when $Q \succeq 0$ results in an relaxation equivalent to the recent optimal diagonal splitting approach of [28]. We first characterize all valid inequalities for $\mathbf{conv}(S)$ that involve only x, $\mathbf{diag}(X)$ and z.

Theorem 2. *A point* $(\bar{x}, \bar{z}, \bar{X})$ *satisfies all valid inequalities* $\sum_{i=1}^n b_i X_{ii} + \alpha^T x + \gamma \le \delta^T z$ *for* $\mathbf{conv}(S)$ *if and only if it is in the convex set*

$$\mathbf{P} := \left\{ (x,z,X) \middle| \begin{array}{l} 0 \le X_{ii} \le x_i \le z_i \le 1, \\ X_{ii} z_i \ge x_i^2, \forall i = 1,...,n \end{array} \right\}.$$

Proof. Note that the definition of $\mathbf{P}$ involves only x, z and $\mathbf{diag}(X)$. For all $i = 1, \ldots, n$, since $X_i i \ge 0$ and $z_i \ge 0$, the second-order-cone representable constraints $X_{ii} z_i \ge x_i^2$ are can be replaced by their (infinite number of) linearized inequalities. At point $(\hat{x}_i, \hat{X}_{ii}, \hat{z}_i)$ such that $\hat{X}_{ii}\hat{z}_i = \hat{x}_i^2$ and $0 \le \hat{x}_i \le \hat{z}_i \le 1$, the linearization is

$$-\hat{z}_i X_{ii} + 2\hat{x}_i x_i \le \hat{X}_{ii} z_i. \tag{7}$$

So if $(\bar{x}, \bar{z}, \bar{X})$ satisfies all $\sum_{i=1}^n b_i X_{ii} + \alpha^T x + \gamma \le \delta^T z$ that are valid for $\mathbf{conv}(S)$, it must be in $\mathbf{P}$.

Next we claim that if $\sum_{i=1}^n b_i X_{ii} + \alpha^T x + \gamma \le \delta^T z$ is valid for $\mathbf{conv}(S)$, then $\gamma \le \min\{\delta^T z - \sum_{i=1}^n b_i x_i^2 - \alpha^T x | 0 \le x_i \le z_i \in \{0,1\}, \forall i\}$. Define $\gamma_i = \min\{\delta_i z_i - b_i x_i^2 - \alpha_i x_i | 0 \le x_i \le z_i \in \{0,1\}\}$, we have $\gamma \le \sum_{i=1}^n \gamma_i$, and each disaggregated inequality $b_i X_{ii} + \alpha_i x_i + \gamma_i \le \delta_i z_i$ is valid for $\{(x_i, z_i, x_i^2) | 0 \le x_i \le z_i \in \{0,1\}\}$. By the convex hull characterization of the latter set (for example [17]), such a disaggregated inequality is valid for $\mathbf{P}$. Therefore $\sum_{i=1}^n b_i X_{ii} + \alpha^T x + \gamma \le \delta^T z$ is also valid for $\mathbf{P}$.

The inequalities $X_{ii} z_i \ge x_i^2$ are called *perspective constraints* in the literature [16–18]. In these works, the variables X_{ii} are introduced to represent x_i^2. For fixed i, in the space of (x_i, z_i, X_{ii}), the lower convex envelope of the feasible set $\{(0,0,0)\} \cup \{(x_i, 1, x_i^2) | 0 \le x_i \le 1\}$ is

$$\tilde{X}_{ii}(z_i, x_i) = \begin{cases} \frac{x_i^2}{z_i}, & 0 \le x_i \le z_i \le 1, z_i \neq 0 \\ 0, & x_i = z_i = 0. \end{cases}$$

So we see that $X_{ii} \geq \tilde{X}_{ii}(z_i, x_i)$ is equivalent to $X_{ii}z_i \geq x_i^2$ with additional restriction $0 \leq X_{ii} \leq x_i \leq z_i \leq 1$.

It is shown, for example in [17], that if the nonlinear functions are appropriately separable (in our context, that there are no off-diagonal entries of X appearing in the objective or constraints), employing perspective constraints improves the solution time significantly for convex MINLPs. For the case of non-separable quadratic programs, one approach is to extract a separable part from the objective function, and apply the perspective constraints on this separable part. We briefly describe this procedure here and show how it is related with the simplest class of lifted-concave-QPB inequalities.

Let ζ denote the optimal value of (1) with $Q \succeq 0$. A method to strengthen the continuous relaxation of (1) proposed by [16] is to find a diagonal matrix D with $D_{ii} \geq 0\ \forall i$ and $Q - D \succeq 0$, and to solve the diagonally-split convex (perspective) relaxation

$$\zeta_{PR}(D) := \min_{p,x,z} \left\{ x^T(Q-D)x + \sum_{i=1}^n p_i + q^T x + c^T z \,\middle|\, \begin{array}{l} Ax + Bz \leq b, p_i z_i \geq D_{ii} x_i^2 \\ 0 \leq x_i \leq z_i \leq 1, \forall i \end{array} \right\}.$$

The constraints $p_i z_i \geq D_{ii} x_i^2$ come from the fact that the function $f(x_i, z_i) = \frac{D_{ii}x_i^2}{z_i}$ (if we define $f(0,0) = 0$) is the lower convex envelope of set $\{(0,0)\} \cup \left\{(D_{ii}x_i^2, 1) \mid 0 \leq x_i \leq 1\right\}$ in the space of (x_i, z_i). The matrix D can be chosen to be $\lambda_{\min} I$ if Q is positive definite with $\lambda_{\min} > 0$ as its minimum eigenvalue, or D can be obtained from the solution of a semidefinite program that seeks to maximize its trace. The work [16] also illustrates that this approach improves the performance of standard commercial solvers by several orders of magnitude on some portfolio optimization problems. In [16], the convex constraints $p_i z_i \geq D_{ii} x_i^2$ are used to generate linear cutting planes (perspective cuts) like (7).

An alternative way of constructing a tight relaxation is to use SDP. The standard semidefinite relaxation for (1) is

$$\zeta_{SDP} := \min \left\{ Q \bullet X + q^T x + c^T z \,\middle|\, \begin{array}{l} X \succeq xx^T, Ax + Bz \leq b, \\ 0 \leq x_i \leq z_i \leq 1, \forall i \end{array} \right\}, \tag{8}$$

and it is easy to show that the bound obtained from (8) is equal to the bound obtained from the continuous relaxation of (**??**). However, if we strengthen (1) by adding the perspective constraints as in Theorem 2, we obtain a semidefinite relaxation which is no worse than $\zeta_{PR}(D)$ with **any** valid splitting $Q = D + (Q - D)$. Specifically, if we define

$$\zeta_{SDP/PR} := \min \left\{ Q \bullet X + q^T x + c^T z \,\middle|\, \begin{array}{l} X \succeq xx^T, Ax + Bz \leq b, \\ X_{ii} z_i \geq x_i^2, 0 \leq x_i \leq z_i \leq 1, \forall i \end{array} \right\}, \tag{9}$$

then we have the following proposition.

Proposition 7 *For all diagonal $D \succeq 0$ and $Q - D \succeq 0$, $\zeta \geq \zeta_{SDP/PR} \geq \zeta_{PR}(D)$.*

Proof. It is straightforward to see $\zeta \geq \zeta_{SDP/PR}$. Suppose $(\bar{x}, \bar{X}, \bar{z})$ is an optimal solution to (9), then for any nonnegative diagonal D such that $Q - D \succeq 0$,

$$\begin{aligned}\zeta_{SDP/PR} &= Q \bullet \bar{X} + q^T \bar{x} + c^T \bar{z} = D \bullet \bar{X} + (Q-D) \bullet \bar{X} + q^T \bar{x} + c^T \bar{z} \\ &\geq \sum_{i:\bar{z}_i>0} D_{ii} \frac{\bar{x}_i^2}{\bar{z}_i} + \bar{x}^T (Q-D)\bar{x} + q^T \bar{x} + c^T \bar{z} \ \geq \ \zeta_{PR}(D).\end{aligned}$$

The first inequality is due to the fact that $\bar{X}_{ii}\bar{z}_i \geq \bar{x}_i^2$ and $\bar{X} \succeq \bar{x}\bar{x}^T$, and last one is by definition of $\zeta_{PR}(D)$.

Further, if under some mild conditions, we can illustrate that there exists an "optimal" D^* such that $\zeta_{SDP/PR} = \zeta_{PR}(D^*)$. This result can be seen as a more natural derivation of the (slightly generalized) main result in [28].

Proposition 8 *Suppose at least one of the following two conditions are satisfied,*

1. $\exists \bar{x}, \bar{z}$ *such that* $A\bar{x} + B\bar{z} < b, 0 < \bar{x}_i < \bar{z}_i < 1, \forall i = 1, \ldots, n$ *(Slater Condition);*
2. Q *is positive definite.*

Let $\left(\hat{y}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{s}, \hat{v}, \hat{W}, \hat{\lambda}, \hat{\mu}, \hat{\tau}\right)$ *be an optimal solution to the following semidefinite optimization*

$$\begin{aligned}\zeta^D_{SDP/PR} := \max \ & -b^T y - s - e^T \tau \\ s.t. \ & Q - \mathbf{diag}(\alpha) = W \\ & c + A^T y = 2\gamma + 2v + \lambda - \mu \\ & d + B^T y = \beta + \mu - \tau \\ & \begin{pmatrix} s & v^T \\ v & W \end{pmatrix} \succeq 0, \begin{pmatrix} \alpha_i & \gamma_i \\ \gamma_i & \beta_i \end{pmatrix} \succeq 0, \forall i = 1, ..., n, \\ & y, \lambda, \mu, \tau \in \mathbb{R}^n_+,\end{aligned}$$

then $\zeta_{PR}(\mathbf{diag}(\hat{\alpha})) = \zeta_{SDP/PR} = \zeta^D_{SDP/PR}$.

Proof. The proof is given in the appendix.

Two remarks are in order. First, Proposition 7 and 8 are relevant to results for the so called QCR method [6, 7]. The QCR method aims to **convexify** nonconvex quadratic programs by adding terms which do not change the optimal value, for example by adding a constant multiple of $x_i^2 - x_i$ if x_i is binary, or $(a^T x - b)^2$ if $a^T x = b$ is a valid constraint. The diagonal splitting approach works in the opposite manner. One starts with a convex objective, extracts a separable part while maintaining the convexity, and strengthen the separable terms using perspective constraints. It is interesting that in both cases, the optimal reformulation parameters can be found by solving an SDP. Second, as suggested by Kurt Anstreicher (personal communication), the inequalities $X_{ii} z_i \geq x_i^2$ are implied by the standard doubly nonnegative relaxations [9, 10] for (1).

5 Tractability of Separation of Lifted Concave QPB Inequalities via Simultaneous Lifting

In this section we show that if the number of binary variables appearing in the inequality ($\mathbf{Card}(\delta)$) is fixed, then separation for lifted-concave-QPB inequalities (**??**) can be accomplished by solving a semidefinite programming problem of size polynomial in n. Key to showing this result is a "dual" result to Proposition 2, which gives a direct characterization of all concave **QPB** inequalities.

Lemma 1. *An inequality $B \bullet X + \alpha^T x + \gamma \le 0$ is a concave* **QPB** *inequality if and only if (B, α, γ) is in the following set $\mathcal{V}_n$:*

$$\mathcal{V}_n := \left\{ (B, \alpha, \gamma) \left| \begin{array}{l} \begin{pmatrix} s & v^T \\ v & -B \end{pmatrix} \succeq 0, \;\; \mu - 2v + \lambda = \alpha \\ -s - \mu^T e \ge \gamma, \, v \in \mathbb{R}^n, \, \lambda, \mu \in \mathbb{R}^n_+, \, s \ge 0 \end{array} \right. \right\}.$$

Proof. Note $B \bullet X + \alpha^T x + \gamma \le 0$ is a concave **QPB** inequality if and only if the following optimization (P) has nonpositive optimal objective value, where (D) is the associated dual problem.

$$\begin{aligned} \max_{0 \le x \le e} \quad & B \bullet X + \alpha^T x + \gamma \\ s.t., \quad & \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \succeq 0 \end{aligned} \qquad \text{(P)}$$

$$\begin{aligned} \min_{\lambda, \mu \in \mathbb{R}^n_+} \quad & \gamma + s + \mu^T e, \\ s.t., \quad & \alpha = \mu - 2v - \lambda \\ & \begin{pmatrix} s & v^T \\ v & -B \end{pmatrix} \succeq 0 \end{aligned} \qquad \text{(D)}$$

Note the primal problem satisfies the Slater condition. Hence strong duality holds and the conclusion easily follows.

Note that $B \bullet X + \alpha^T x + \gamma \le \delta^T z$ is a valid lifted concave inequality and $\mathbf{Card}(\delta) \le k$ if and only if for all $I \subseteq \{1, ..., n\}$, $|I| \ge n-k$, $(B_{[I,I]}, \alpha_I, \gamma - \delta^T e_I) \in \mathcal{V}_{|I|}$, where $B_{[I,I]}$, α_I are the corresponding principal submatrice and subvector, and e_I is a vector with ones at indices in I and zeros elsewhere. Then for fixed k, the separation problem of all lifted concave inequalities with $\mathbf{Card}(\delta) \le k$ can be written as an SDP of polynomial size in n. Note that in general the SDP size is of $O(n^k)$.

At the end of this paper, we provide a small computational example to illustrate that, although the simplest lifted concave inequalities (perspective cuts) are implied by the DNN relaxation, in general lifted concave inequalities are not. (Actually this is not surprising in light of Proposition 6). Also this example seems to suggest the importance of lifted concave inequalities with $\mathbf{rank}(B)$ small.

Example 1 (Non-dominance by doubly nonnegative relaxation). We consider the following convex quadratic program with binary indicators

$$\begin{aligned} \min_{x \in [0,1]^3} \quad & x^T Q x + c^T x + d^T z \\ s.t. \quad & 0 \le x_i \le z_i, z_i \in \{0, 1\}, i = 1, 2, 3, \end{aligned}$$

where

$$Q = \begin{pmatrix} 4.4 & 3.1 & -4.2 \\ 3.1 & 3.0 & -3.2 \\ -4.2 & -3.2 & 4.6 \end{pmatrix}, c = \begin{pmatrix} -1.4 \\ -1.4 \\ 0.1 \end{pmatrix}, d = \begin{pmatrix} 0.4 \\ 0.2 \\ 0.5 \end{pmatrix}$$

One can verify that the optimal value is 0 and the optimal solution is $x = z = 0$. The DNN relaxation [9] (solved by using Yalmip [21] with CSDP [8]) yields a lower bound that equals approximately $-3.89E - 2$. Then we employ the SDP-based separation procedure based on Lemma with $k = 3$ to generate a valid lifted concave inequality, and then resolve the strengthened DNN relaxation. The lower bound is improved to the exact optimal value 0 (with accuracy about 10^{-10}) after three rounds. This verifies Proposition 6. Also it is worth noting that the eigenvalues of B matrices in three cuts are

$$[0.0000, 0.0000, -0.5492], [0.0000, -0.0469, -0.6526], [0.0000, 0.0000, -0.7511],$$

respectively, i.e., all of the B matrices are close to rank-1.

6 Discussion and Future Work

Results in this extended abstract leave some interesting open questions that we hope to address in future work. First, note for the set **QPB**, we may assume that all concave inequalities have $\mathbf{rank}(B) \leq 1$. A natural question is the extent to which this result is true for $\mathbf{conv}(S)$. Example 1 suggests that lifted concave-QPB inequalities with low rank of B may be more important than those with high rank. Next, can we design effective separation heuristic algorithms for lifted concave-QPB inequalities, especially when B has low rank? Last but not least, does the lifted concave approach motivate "projected formulations" where one derives valid inequalities using only $O(n)$ number of variables, as what has been done in [25] for **QPB**?

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Appendix

Proof of Proposition 1.

Proof. To show $\mathbf{conv}(S)$ is full-dimensional, we enumerate the following affinity independent points in S. All entries we do not mention are assumed to be 0.

1. $(x, z, X) = 0$;
2. $z_i = 1, x = 0, X = 0$, for $i = 1, ..., n$;
3. $z_i = 1, x_i = 1, X_{ii} = 1$, for $i = 1, ..., n$;
4. $z_i = 1, x_i = 0.5, X_{ii} = 0.25$, for $i = 1, ..., n$;
5. $z_i = z_j = 1, x_i = x_j = X_{ii} = X_{ij} = X_{jj} = 1$, for $1 \le i < j \le n$.

The above $\frac{n(n+1)}{2} + 2n + 1$ points are affinely independent. Therefore $\mathbf{conv}(S)$ is full-dimensional. Because $S \subseteq S^{\succeq}$, $\mathbf{conv}(S^{\succeq})$ is also full-dimensional.

Since every extreme point of $\mathbf{conv}(S)$ is in S, to show the second result, it suffices to show every point in S is extremal in $\mathbf{conv}(S)$. If otherwise, there exists $\hat{x}$ and $\left\{x^{(1)}, \cdots, x^{(K)}\right\} \in \mathbb{R}^n$ such that $\hat{x}\hat{x}^T = \sum_{j=1}^K \lambda_j x^{(j)} x^{(j)T}$, where $\lambda_j \ge 0$ and $\sum_{j=1}^K \lambda_j = 1$. This contradicts with the extremal characterization of the positive semidefinite cone. Therefore the set of extreme points for $\mathbf{conv}(S)$ is exactly S.

For the last result, take a point $(\bar{x}, \bar{z}, \bar{X}) \in \mathbf{conv}(S)$ and $\tilde{X} \succeq 0$, and let $(\bar{x}, \bar{z}, \bar{X}) = \sum_j \lambda_j (x^{(j)}, z^{(j)}, x^{(j)} x^{(j)T})$ be the convex combination of points in S, then $(\bar{x}, \bar{z}, \bar{X} + \tilde{X}) = \lambda_1 (x^{(j)}, z^{(j)}, x^{(j)} x^{(j)T} + \tilde{X}) + \sum_{j>1} \lambda_j (x^{(j)}, z^{(j)}, x^{(j)} x^{(j)T}) \in \mathbf{conv}(S^{\succeq})$. So

$$\mathbf{conv}(S^{\succeq}) \supseteq \mathbf{conv}(S) + \left\{(0, 0, X) \in \mathbb{R}^{2n + \frac{n(n+1)}{2}}, X \succeq 0\right\}.$$

To show the other direction, note every point $(x, z, X) \in S^{\succeq}$ can be written as $(x, z, xx^T) + (0, 0, X - xx^T)$, hence is in $\mathbf{conv}(S) + \left\{(0, 0, X) \in \mathbb{R}^{2n + \frac{n(n+1)}{2}}, X \succeq 0\right\}$.

Proof of Proposition 4.

Proof. Suppose that $\exists (\bar{x}, \bar{X})$ such that $(\bar{x}, \mathbf{ADiag}(\bar{X})) \in \mathbf{BQP}$ and $\bar{X}_{ii} \le \bar{x}_i \forall i = 1, \ldots, n$. By the properties of projection, there then exist $y_1, y_2, \ldots, y_K \in \{0, 1\}^n$ such that

$$(\bar{x}, \bar{X} - \mathbf{Diag}(\bar{X}) + \mathbf{Diag}(\bar{x})) = \sum_{k=1}^K \lambda_k (y_k, y_k y_k^T),$$

where $\lambda_k \ge 0, \forall k = 1, \ldots, K$, $\sum_{k=1}^K \lambda_k = 1$, and $\bar{X} - \mathbf{Diag}(\bar{X}) + \mathbf{Diag}(\bar{x})$ is the matrix $\bar{X}$ with its diagonal replaced by entries in $\bar{x}$. Then $B \bullet \bar{X} + \alpha^T \bar{x} + \gamma$ equals

$$\begin{aligned} & B \bullet (\bar{X} - \mathbf{Diag}(\bar{X}) + \mathbf{diag}(\bar{x})) + \alpha^T \bar{x} + \gamma + \sum_{i=1}^n B_{ii} (\bar{X}_{ii} - \bar{x}_i) \\ \le\ & B \bullet \left(\sum_{k=1}^K \lambda_k y_k y_k^T\right) + \alpha^T \left(\sum_{k=1}^K \lambda_k y_k\right) + \gamma \le 0. \end{aligned}$$

The first inequality is because $\bar{X}_{ii} \leq \bar{x}_i$ and $B_{ii} \geq 0$, and the second inequality is because $B \bullet X + \alpha^T x + \gamma \leq 0$ is valid for **QPB**, hence valid for $(y_k, y_k y_k^T) \forall k = 1, \ldots, K$.

The opposite direction of the proof is easy because **BQP** equals a projection of **QPB**, so any inequality from $(x, \mathbf{ADiag}(X)) \in \mathbf{BQP}$ is a posdiag inequality for **QPB** as all diagonal coefficients are zeros.

Proof of Proposition 5

Proof. By Proposition 1, all $X \succeq 0$ defines a recession direction for $\mathbf{conv}(S^{\succeq})$. From this, and the fact that x and z are bounded in $\mathbf{conv}(S^{\succeq})$, if $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ with $B_{ii} \geq 0 \ \forall i = 1, \ldots, n$ is valid for $\mathbf{conv}(S^{\succeq})$, then we must have $B \preceq 0$. Together with $B_{ii} \geq 0$, $\forall i = 1, \ldots, n$, it follows that $B = 0$. Further, if $\alpha^T x + \gamma \leq \delta^T z$ is valid for $\mathbf{conv}(S^{\succeq})$, then it is valid for $\{(x, z) | 0 \leq x \leq z \leq 1\}$, which is the projection of $\mathbf{conv}(S^{\succeq})$ onto the space of (x, z). The other direction is trivial.

Proof of Claim 3.

Proof. For any triplet (x, xx^T, z) such that $0 \leq x_i \leq z_i \in \{0, 1\}, \forall i$, there is a triplet $(\tilde{x}, \tilde{x}\tilde{x}^T, \tilde{z})$ such that $\|x - \tilde{x}\|$ is arbitrarily small, $0 \leq \tilde{x}_i \leq \tilde{z}_i \in \{0, 1\}, \forall i$, and $\tilde{z}_i = z_i \ \forall i$ such that $\delta_i \geq 0$ and $\tilde{z}_i = 1 \ \forall i$ such that $\delta_i < 0$. Therefore if (x, xx^T, z) violates (6), i.e. if $B \bullet xx^T + \alpha^T x + \gamma > \sum_{i:\delta_i \geq 0} \delta_i z_i + \sum_{i:\delta_i < 0} \delta_i$, it must be that $B \bullet \tilde{x}\tilde{x}^T + \alpha^T \tilde{x} + \gamma > \delta^T \tilde{z}$, so $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ was not valid for $\mathbf{conv}(S)$. Note that (6) is a valid inequality with all coefficients of z nonnegative. Of course, if $(\bar{x}, \bar{X}, \bar{z})$ satisfies (6) then it satisfies $B \bullet \bar{X} + \alpha^T \bar{x} + \gamma \leq \delta^T \bar{z}$.

Proof of Proposition 6.

Proof. Let $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ be a valid inequality for $\mathbf{conv}(S)$ and $B \preceq 0$. Because of Proposition 1, $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ is also valid for $\mathbf{conv}(S^{\succeq})$. To prove the converse, if $B \bullet X + \alpha^T x + \gamma \leq \delta^T z$ is a valid inequality for $\mathbf{conv}(S^{\succeq})$, because $\mathbf{conv}(S^{\succeq})$ has the recession cone $\left\{(0, 0, X) \in \mathbb{R}^{2n + \frac{n(n+1)}{2}}, X \succeq 0\right\}$, we must have $B \preceq 0$.

Proof of Proposition 8.

Proof. In (9) we may rewrite $X \succeq xx^T$ and $X_{ii} z_i \geq x_i^2$ as $\begin{pmatrix} 1 & x \\ x & X \end{pmatrix} \succeq 0$ and $\begin{pmatrix} X_{ii} & x_i \\ x_i & z_i \end{pmatrix} \succeq 0$. Then by introducing dual variables $\begin{pmatrix} s & v^T \\ v & W \end{pmatrix}$ and $\begin{pmatrix} \alpha_i & \gamma_i \\ \gamma_i & \beta_i \end{pmatrix}$ for them respectively, and y for $Ax + Bz \leq b$, λ, μ, τ for $0 \leq x \leq z \leq e$, it is straightforward to verify that our optimization problem is the dual problem of (9). Also condition 1 implies (9) is strictly feasible, and condition 2 implies the dual SDP is strictly feasible. Hence by strong duality $\zeta_{SDP/PR} = \zeta_{SDP/PR}^D$.

Now we show that $\zeta_{PR}(\mathbf{diag}(\hat{\alpha})) = \zeta_{SDP/PR}$. By Proposition 7 it suffices to show $\zeta_{PR}(\mathbf{diag}(\hat{\alpha})) \geq \zeta^D_{SDP/PR}$. Assume $(\bar{x}, \bar{X}, \bar{z}, \bar{p})$ is feasible in (7), then

$$\begin{aligned}
-b^T\hat{y} - \hat{s} - e^T\hat{\tau} &\leq -(A\bar{x} + B\bar{z})^T y - \hat{s} - e^T\hat{\tau} \\
&\leq -\bar{x}^T(2\hat{\gamma} + 2\hat{v} + \hat{\lambda} - \hat{\mu} - c) - \bar{z}^T(\hat{\beta} + \hat{\mu} - \hat{\tau} - d) - s - e^T\hat{\tau} \\
&\leq c^T\bar{x} + d^T\bar{z} + \bar{x}^T(Q - \mathbf{diag}(\alpha))\bar{x} \\
&\quad -\bar{x}^T(2\hat{\gamma} + \hat{\lambda} - \hat{\mu} - c) - \bar{z}^T(\hat{\beta} + \hat{\mu} - \hat{\tau} - d) - e^T\hat{\tau} \\
&\leq c^T\bar{x} + d^T\bar{z} + \bar{x}^T(Q - \mathbf{diag}(\alpha))\bar{x} + \sum_i \bar{p}_i - \bar{x}^T(\hat{\lambda} - \hat{\mu}) \\
&\quad -\bar{z}^T(\hat{\mu} - \hat{\tau}) - e^T\hat{\tau} \\
&\leq c^T\bar{x} + d^T\bar{z} + \bar{x}^T(Q - \mathbf{diag}(\alpha))\bar{x} + \sum_i \bar{p}_i
\end{aligned}$$

The second inequality is because $A^T\hat{y} = 2\hat{\gamma}+2\hat{v}+\hat{\lambda}-\hat{\mu}-c$ and $B^T\hat{y} = \hat{\beta}+\hat{\mu}-\hat{\tau}-d$. The third inequality is because $\begin{pmatrix} 1 & \bar{x} \\ \bar{x} & \bar{x}\bar{x}^T \end{pmatrix} \bullet \begin{pmatrix} \hat{s} & \hat{v}^T \\ \hat{v} & \hat{W} \end{pmatrix} \geq 0 \Leftrightarrow -\hat{s} - 2\hat{v}^T\bar{x} \leq \bar{x}^T(Q - \mathbf{diag}(\hat{\alpha}))\bar{x}$. The fourth is because when $\hat{\alpha}_i \neq 0$, $\begin{pmatrix} \frac{\bar{p}_i}{\hat{\alpha}_i} & \bar{x}_i \\ \bar{x}_i & \bar{z}_i \end{pmatrix} \bullet \begin{pmatrix} \hat{\alpha}_i & \hat{\gamma}_i \\ \hat{\gamma}_i & \hat{\beta}_i \end{pmatrix} \geq 0 \Leftrightarrow -\hat{\beta}_i\bar{z}_i - 2\hat{\gamma}_i\bar{x}_i \leq \bar{p}_i$, and when $\hat{\alpha}_i = 0$, $\hat{\gamma}_i = 0$, $-\hat{\beta}_i\bar{z}_i \leq 0 \leq \bar{p}_i$. The last inequality is because of nonnegativity of variables and $0 \leq \bar{x} \leq \bar{z} \leq e$.