PERSPECTIVE RELAXATION OF MINLPS WITH INDICATOR VARIABLES

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ABSTRACT. We study mixed integer nonlinear programs (MINLP) that are driven by a collection of indicator variables where each indicator variable controls a subset of the decision variables. An indicator variable, when it is "turned off", forces some of the decision variables to assume a fixed value, and, when it is "turned on", forces them to belong to a convex set. Most of the integer variables in known MINLP problems are of this type.

We first study a mixed integer set defined by a single separable quadratic constraint and a collection of variable upper and lower bound constraints. This is an interesting set that appears as a substructure in many applications. We present the convex hull description of this set. We then extend this to produce an explicit characterization of the convex hull of the union of a point and a bounded convex set defined by analytic functions. Further, we show that for many classes of problems, the convex hull can be expressed via conic quadratic constraints, and thus relaxations can be solved via second-order cone programming. Our work is closely related with the earlier work of Ceria and Soares (1996) as well as recent work by Frangioni and Gentile (2006) and, Aktürk, Atamtürk and Gürel (2007).

Finally, we apply our results to develop tight formulations of mixed integer nonlinear programs in which the nonlinear functions are separable and convex and in which indicator variables play an important role. In particular, we present strong computational results with two applications – quadratic facility location and network design with congestion – that show the power of the reformulation technique.

Keywords: Mixed-integer nonlinear programming; perspective functions;

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1. INTRODUCTION

In this work, we study mixed integer nonlinear programs (MINLP) that are driven by a collection of indicator variables where each indicator variable controls a subset of the decision variables. In particular, we are interested in MINLPs where an indicator variable, when it is "turned off", forces some of the decision variables to assume a fixed value, and, when it is "turned on", forces them to belong to a convex set. We call such programs *indicator-induced {0,1}-mixed integer nonlinear programs*.

A generic indicator-induced {0-1}-MINLP can be written as

$$\min_{(x,z)\in X\times(Z\cap\mathbb{B}^p)} \{c^T x + d^T z \mid g_j(x,z) \le 0\ \forall j \in M,\ (x_{V_i}, z_i) \in S_i\ \forall i \in I\} \tag{1}$$

where z are the indicator variables, x are the continuous variables and x_{V_i} denotes the collection of continuous variables (i.e. $x_j,\ j \in V_i$) controlled by the indicator variable z_i. Sets $X \subseteq \mathbb{R}^n$ and $Z \subseteq \mathbb{R}^p$ are polyhedral sets of appropriate dimension and S_i is the set of points that satisfy all constraints associated with the indicator variable z_i:

$$S_i \stackrel{\text{def}}{=} \left\{ (x_{V_i}, z_i) \in \mathbb{R}^{|V_i|} \times \mathbb{B} \;\middle|\; \begin{array}{ll} x_{V_i} = \hat{x}_{V_i} & \text{if } z_i = 0 \\ x_{V_i} \in \Gamma_i & \text{if } z_i = 1 \end{array} \right\},$$

where

$$\Gamma_i \stackrel{\text{def}}{=} \{x_{V_i} \in \mathbb{R}^{|V_i|} \mid f_j(x_{V_i}) \le 0\ \forall j \in C_i,\ u_k \ge x_k \ge \ell_k\ \forall k \in V_i\}.$$

is bounded for all $i \in I$. The objective function in (1) is assumed to be linear without loss of generality (if necessary, an additional variable can be used to move the nonlinearity from the objective function to the constraint set.)

In this paper we study the convex hull description of the sets S_i when Γ_i is a convex set. Note that Γ_i can be convex even when some of the f_j defining it are non-convex. Let $S_i^c = conv(S_i)$. Using S_i^c, one can write a "tight" continuous relaxation of (1)

$$\min_{(x,z)\in X\times Z} \{c^T x + d^T z \mid g_j(x,z) \le 0\ \forall j \in M,\ (x_{V_i}, z_i) \in S_i^c\ \forall i \in I\} \tag{2}$$

where S_i in (1) is replaced by its convex hull and integrality requirement on z is dropped. We assume that Z already contains bound constraints for z. We call (2), the *perspective relaxation* of (1) as description of S_i^c involves *perspective* functions which we discuss later. We also present computational results and show that (2) indeed gives a strong relaxation when applied to a number of problems. We also show that in some cases, S_i^c is representable as a second-order cone and this improves computational effectiveness of our approach even further.

Indicator-induced MINLPs can be used to model many interesting problems. Two applications that we study in detail in this paper are: (i) the quadratic-cost uncapacitated facility location problem recently studied by Günlük et al. [14], and, (ii) network design problem under queuing delay, first discussed by Boorstyn and Frank [8]. For other examples see[20, 6, 15] for portfolio optimization problems; or, Aktürk et al. [1] for a job-scheduling problem with controllable processing times. In addition, certain classes of unit commitment problems for electrical power generation can be formulated as indicator-induced MINLPs.

There has been some recent work on generating strong relaxations for convex MINLPs. One line of work has been on extending general classes of cutting planes from mixed integer linear programs. Specifically, Stubbs and Mehrotra [21] explain how the disjunctive cutting planes of Balas et al. [3] can be applied for MINLP, Cezik and Iyengar [11] extend the Gomory cutting plane procedure [13], and Atamtürk and Narayanan [2] extend the mixed integer rounding procedure of Nemhauser and Wolsey [18] to conic mixed integer programs. A second line of work has focused on generating problem specific cutting planes, for example see Günlük et al. [14] for different families of inequalities for a quadratic cost facility location problem. In some cases these inequalities can be used to strengthen the perspective relaxation even further.

There are two recent papers that are closely related with our work. Frangioni and Gentile [12] have introduced a class of linear inequalities called *perspective cuts* for indicator-induced MINLPs. As we discuss later, perspective cuts are essentially outer approximation cuts for S_i^c and therefore the perspective relaxation

(2) can be viewed as implicitly including all (infinitely many) perspective cuts to a straightforward relaxation of (1). Very recently, Aktürk et al. [1] independently gave a strong characterization of S_i^c when $\Gamma_i = \{x \in \mathbb{R}^2 \mid x_1^t - x_2 \le 0,\ u \ge x_1, x_2 \ge 0\}$ for $t \ge 1$. They use this characterization in an algorithm for solving some classes of nonlinear machine scheduling problems.

2. A Quadratic Set with Variable Bounds

In this section we present a convex hull description of the following set

$$Q = \Big\{ w \in \mathbb{R},\ x \in \mathbb{R}^n_,\ z \in \times\mathbb{B}^n : w \ge \sum_{i=1}^n q_i x_i^2,\ u_i z_i \ge x_i \ge l_i z_i,\ x_i \ge 0,\ i = 1, 2, \dots, n \Big\}$$

where $q_i \in \mathbb{R}_+$ and $u_i, l_i \in \mathbb{R}$ for all $i = 1, 2, \dots, n$. We then use this insight to define the convex hull of more complicated mixed integer nonlinear sets. Set Q appears in a number of non-linear mixed-integer programs as a substructure and we present some examples of this in Section 4. To our knowledge, the first convex hull description of Q was stated without proof in the unpublished Ph.D. thesis of Stubbs [22].

2.1. A Simple Set.

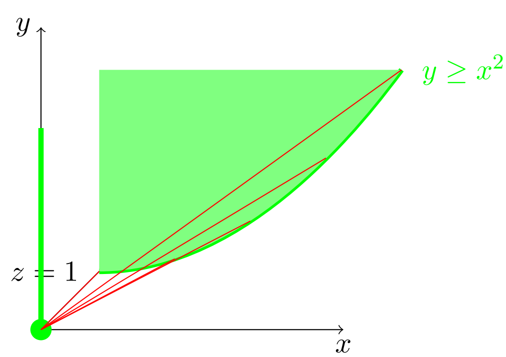
To understand the set Q, we first study a simpler mixed-integer set with only 3 variables, which can be obtained by setting $n = 1$ and $q_1 = 1$. Let

$$S = \Big\{ (x, y, z) \in \mathbb{R}^2 \times \mathbb{B} \ :\ y \ge x^2,\ uz \ge x \ge lz,\ x \ge 0 \Big\}$$

where $u, l \in \mathbb{R}$. We next show that the convex hull of S is given by

$$S^c \ = \ \big\{ (x, y, z) \in \mathbb{R}^3 \ :\ yz \ge x^2,\ uz \ge x \ge lz,\ 1 \ge z \ge 0,\ x, y \ge 0 \big\}.$$

FIGURE 1: The set S^c



Note that even though $yz \ge x^2$ is not a convex constraint (as its Hessian is not positive semi-definite), it still defines a convex set in $\mathbb{R}^3_+$.

Lemma 1. $\mathrm{conv}(S) \ = \ S^c$.

Geometrically, the set S^c consists of all points that lie above a line segment connecting the origin to the point $(t, t^2, 1)$ for each $t \ \ge \ 0$. The set is shown in Figure 2.1.

2.2. An extended formulation.

Consider the following extended formulation of Q

$$\bar{Q} \ = \ \Big\{ w \in \mathbb{R},\ x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \mathbb{R}^n \ :\ w \ge \sum_i q_i y_i,\ (x_i, y_i, z_i) \in S_i,\ i = 1, 2, \dots, n \Big\}$$

where S_i has the same form as the set S discussed in the previous section except the bounds u and l are replaced with u_i and l_i. Note that if $(w, x, y, z) \in \bar{Q}$ then $(w, x, z) \in Q$, and therefore $\mathrm{proj}_{(w,x,z)}(\bar{Q}) \subseteq Q$. On the other hand, for any $(w, x, z) \in Q$, letting let $y_i' = x_i^2$ gives a point $(w, x, y', z) \in \bar{Q}$. Therefore, $\bar{Q}$ is indeed an extended formulation of Q, or, in other words, $Q = \mathrm{proj}_{(w,x,z)}(\bar{Q})$.

Before we present a convex hull description of $\bar{Q}$ we first define some basic properties of mixed-integer sets which are not necessarily polyhedral. Using these definitions, we then show some elementary observations which are known for polyhedral sets.

Definition 1. Given a closed set $P \subset \mathbb{R}^n$, point $p \in P$ is called an *extreme point* of P if it can not be represented as $p = 1/2 p_1 + 1/2 p_2$ for $p_1, p_2 \in P$, $p_1 \ne p_2$. Set P is called *pointed* if it has extreme points.

Definition 2. A closed, pointed set $P \subset \mathbb{R}^n$ is called *integral* with respect to a subset of the indices $I \subseteq \{1, \dots, n\}$ if for any extreme point $p \in P$, $p_i \in \mathbb{Z}$ for all $i \in I$.

Lemma 2. *For $i = 1, 2$ let $P_i \subset \mathbb{R}^{n_i}$ be a closed and pointed set which is integral with respect to indices I_i. Furthermore, let $P' = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ x \in P_1, \ y \in P_2\}$.*

(i) P' is integral with respect to $I_1 \cup I_2$.

(ii) $\text{conv}(P') = \{(x, y) \in \mathbb{R}^{n_1+n_2} \ : \ x \in \text{conv}(P_1), \ y \in \text{conv}(P_2)\}$. ■

Lemma 3. *Let $P \subset \mathbb{R}^n$ be a given closed, pointed set and let $P' = \{(w, x) \in \mathbb{R}^{n+1} \ : \ w \geq ax, \ x \in P\}$ where $a \in \mathbb{R}^n$.*

(i) If P is integral with respect to I, then P' is also integral with respect to I.

(ii) $\text{conv}(P') = P''$ where $P'' = \{(w, x) \in \mathbb{R}^{n+1} \ : \ w \geq ax, \ x \in \text{conv}(P)\}$. ■

We are now ready to present the convex hull of $\bar{Q}$. Let

$$\bar{Q}^c = \Big\{w \in \mathbb{R}, \, x \in \mathbb{R}^n_; \, y \in \mathbb{R}^n, \, z \in \times\mathbb{R}^n \ : \ w \geq \sum_i q_i y_i, \, (x_i, y_i, z_i) \in S_i^c, \, i = 1, 2, \ldots, n\Big\}.$$

Lemma 4. *The set $\bar{Q}^c$ is integral with respect to the indices of z variables. Furthermore,* $\text{conv}(\bar{Q}) = \bar{Q}^c$.

Proof. Let $D = \{x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \times\mathbb{R}^n \ : \ (x_i, y_i, z_i) \in S_i, \ i = 1, 2, \ldots, n\}$ so that $\bar{Q} = \{w \in \mathbb{R}, \ x \in \mathbb{R}^n, y \in \mathbb{R}^n, z \in \times\mathbb{R}^n \ : \ w \geq \sum_{i=1}^n q_i y_i, \ (x, y, z) \in D\}$. By Lemma 3, the convex hull of $\bar{Q}$ can be obtained by replacing D with its convex hull in this description. By Lemma 2, this can simply be done by taking convex hulls of S_i's, that is, by replacing S_i with $\text{conv}(S_i)$ in the description of D. Finally, by Lemma 3, $\bar{Q}^c$ is integral. ■

2.3. **Convex hull description in the original space.** Let

$$Q^c \ = \ \Big\{(w, x, z) \in \mathbb{R}^{1+n+n} \ : w \prod_{i \in S} z_i \ \geq \ \sum_{i \in S} (q_i x_i^2 \prod_{l \in S \setminus \{i\}} z_l), \ S \subseteq \{1, 2, \ldots, n\} \qquad (\Pi)$$

$$u_i z_i \geq x_i \geq l_i z_i, \ x_i \geq 0, \ i = 1, 2, \ldots, n\Big\}.$$

Notice that a given point $\bar{p} = (\bar{w}, \bar{x}, \bar{z})$ satisfies inequality (Π) for a particular $S \subseteq \{1, 2, \ldots, n\}$ if an only if one of the following conditions hold: (i) $\bar{z}_i = 0$ for some $i \in S$, or, (ii) if all $z_i > 0$, then $\bar{w} \geq \sum_{i \in S} q_i \bar{x}_i^2 / \bar{z}_i$. Based on this observation we next show that these (exponentially many) inequalities are sufficient to describe the convex hull of Q in the space of the original variables.

Lemma 5. $Q^c = \text{proj}_{(w,x,z)}(\bar{Q}^c)$.

Proof. Let $\bar{p} = (\bar{w}, \bar{x}, \bar{y}, \bar{z}) \in \bar{Q}^c$ and define $S(\bar{p}) = \{i \ : \ z_i > 0\}$. Clearly $u_i \bar{z}_i \geq \bar{x}_i \geq l_i \bar{z}_i$ and $\bar{x}_i \geq 0$ for all $i = 1, 2, \ldots, n$. Furthermore, inequality (Π) is satisfied for all S such that $S \not\subseteq S(\bar{p})$. In addition, notice that, as $q \geq 0$,

$$\bar{w} \ \geq \ \sum_{i \in S(\bar{p})} q_i \bar{y}_i \ \geq \ \sum_{i \in S(\bar{p})} q_i \bar{x}_i^2 / \bar{z}_i \ \geq \ \sum_{i \in S'} q_i \bar{x}_i^2 / \bar{z}_i$$

for all $S' \subseteq S(\bar{p})$. Therefore $\bar{p}$ satisfies inequality (Π) for all S and $\text{proj}_{(w,x,z)}(\bar{Q}^c) \subseteq Q^c$.

Next, let $\bar{p} = (\bar{w}, \bar{x}, \bar{z}) \in Q^c$ be given and let

$$\bar{y}_i = \begin{cases} 0 & \bar{z}_i = 0 \\ \bar{x}_i^2 / \bar{z}_i & \text{otherwise.} \end{cases}$$

It is easy to see that $(\bar{x}_i, \bar{y}_i, \bar{z}_i) \in S_i$ for all $i \in \{1, 2, \ldots, n\}$. Furthermore,

$$\bar{w} \ \geq \ \sum_{i \in S(\bar{p})} q_i \bar{x}_i^2 / \bar{z}_i \ = \ \sum_{i \in S(\bar{p})} q_i \bar{y}_i \ = \ \sum_{i=1}^n q_i \bar{y}_i$$

implying that $(\bar{w}, \bar{x}, \bar{y}, \bar{z}) \in \bar{Q}^c$ and therefore $Q^c \subseteq \text{proj}_{(w,x,z)}(\bar{Q}^c)$. ■

2.4. **SOCP Representation.** A second-order cone constraint is a constraint of the form

$$\|Ax + b\|_2 \leq c^T x + d. \tag{3}$$

The set of points x that satisfy (3) forms a convex set, and efficient and robust algorithms exist for solving optimization problems containing second-order cone constraints [23, 17]. An interesting and important observation from a computational standpoint is that the nonlinear inequalities in the definitions of the sets S^c and $\bar{Q}^c$ can be written as second-order cone constraints. All the nonlinear constraints in the definition S^c and $\bar{Q}^c$ are of the simple form

$$x^2 \leq yz \text{ with } y \geq 0, z \geq 0, \tag{4}$$

and this is algebraically equivalent to the second-order cone constraint

$$\|(2x, y - z)^T\| \leq y + z.$$

Constraints of the form (4) are often called *rotated second order cone* constraints. The computational benefit of dealing with inequalities (4) as second-order cone constraints rather than general nonlinear constraints will be demonstrated in Section 4.1.

3. A Generalization and Connections to Previous Work

We next extend the observations presented in Section 2 to describe the convex hull of a point $\bar{x} \in \mathbb{R}^n$ and a bounded convex set defined by analytic functions. In other words, using an indicator variable $z \in \{0, 1\}$, define $W^0 = \{(x, z) \in \mathbb{R}^{n+1} \ : \ x = \bar{x}, \ z = 0\}$, and

$$W^1 = \{(x, z) \in \mathbb{R}^{n+1} \ : \ f_i(x) \leq 0 \text{ for } i \in I, \ u \geq x - \bar{x} \geq l, \ z = 1\}$$

where $u, l \in \mathbb{R}^n_+$, and $I = \{1, \ldots, t\}$. We are interested in the convex hull of $W = W^1 \cup W^0$. Clearly, both W^0 and W^1 are bounded and W^0 is a convex set. Furthermore, if W^1 is also convex then

$$\mathrm{conv}(W) = \{p \in \mathbb{R}^{n+1} \ : \ p = \alpha p^1 + (1 - \alpha)p^0, \ p^1 \in W^1, \ p^0 \in W^0, \ 1 \geq \alpha \geq 0\}.$$

We next present a description of $\mathrm{conv}(W)$ in the space of original variables.

3.1. **Reformulation in the original space.** To simplify notation we assume that $\bar{x} = 0$ in the remainder of this section. Note that there is no loss of generality as this is an affine transformation. We next write the description of $\mathrm{conv}(W)$ in open form

$$\begin{aligned} \mathrm{conv}(W) \ = \ \Big\{(x, z) \in \mathbb{R}^{n+1} \ : \quad & 1 \ \geq \ \alpha \ \geq \ 0, \\ & x = \alpha x^1 + (1 - \alpha)x^0, \ z = \alpha z^1 + (1 - \alpha)z^0, \\ & x^0 = \bar{x}, \ z^0 = 0, \\ & f_i(x^1) \ \leq 0 \text{ for } i \in I, \ u \geq x^1 - \bar{x} \geq l, \ z^1 = 1 \Big\}. \end{aligned} \tag{XF}$$

The additional variables used in this description can be projected out to obtain a description in the space of the original variables.

Lemma 6. *If W^1 is convex, then $\mathrm{conv}(W) = W^- \cup W^0$, where*

$$W^- \ = \ \Big\{(x, z) \in \mathbb{R}^{n+1} : f_i(x/z) \leq 0 \textit{ for } i \in I, \ uz \geq x \geq lz, 1 \geq z > 0\Big\}.$$

Proof. As x^0, z^0 and z^1 are fixed in (XF), it is possible to substitute out these variables. In addition, as $z = \alpha$ after these substitutions, we can eliminate α. Furthermore, as $x = \alpha x^1 = z x^1$, we can eliminate x^1 by replacing it with x/z provided that $z > 0$. If, on the other hand, $z = 0$, clearly $(x, 0) \in \mathrm{conv}(W)$ if and only if $(x, 0) \in W^0$. ∎

We next show that W^0 is contained in the closure of W^-.

Lemma 7. *For* $1 \geq z > 0$*, let* $Q^c(z) = \Big\{x \in \mathbb{R}^n \,:\, f_i(x/z) \leq 0 \text{ for } i \in I,\ uz \geq x \geq lz\Big\}$*. If all* $f_i(x)$ *are bounded in* $[l,u]$*, then,*

$$lim_{z \to 0^+}\, Q^c(z) = \Big\{x \in \mathbb{R}^n \,:\, x = 0\Big\}$$

Proof. Let $\{z_k\} \subset (0,1)$ be a sequence converging to 0. As, by definition, $Q^c(z) \neq \emptyset$ for $z \in (0,1)$, there exists a corresponding sequence $\{x_k\}$ such that $x_k \in Q^c(z_k)$. Clearly, $uz \geq x_k \geq lz$ and therefore $\{x_k\}$ converges to 0. ∎

Combining the previous lemmas, we obtain the following result.

Corollary 1. $\operatorname{conv}(W) \;=\; \text{closure}(W^-)$.

We would like to emphasize that even when $f(x)$ is a convex function $f_i(x/z)$ may not be convex. However, for $z > 0$ we have

$$f_i(x/z) \leq 0 \;\Leftrightarrow\; z^t f_i(x/z) \leq 0 \tag{5}$$

for any $t \in \mathbb{R}$. In particular, taking $t = 1$ gives $zf_i(x/z)$ which is known to be convex provided that $f(x)$ is convex. We discuss this further in Section 3.2. We also note that if $f(x)$ is SOCP-representable, then $zf_i(x/z)$ is also SOCP-representable and in particular, if W^1 is defined by SOCP-representable functions, then so is $\operatorname{conv}(W)$. We will show the benefits of employing SOC solvers for (non-quadratic) SOC-representable sets in Section 4.2.

When next show that when all $f_i(x)$ that define W^1 are polynomial functions, convex hull of W can be described explicitly.

Lemma 8. *Let* $f_i(x) = \sum_{t=1}^{p_i} c_{it} \prod_{j=1}^{n} x_j^{q_{itj}}$ *for all* $i \in I$*. Let* $q_{it} = \sum_{j=1}^{n} q_{itj}$ *and* $q_i = \max_t\{q_{it}\}$*. If* [what exactly are the conditions we need here?] *all* $f_i(x)$ *are convex and bounded in* $[l,u]$*, then* $\operatorname{conv}(W) = W^c$*, where*

$$W^c \;=\; \Big\{(x,z) \in \mathbb{R}^{n+t+1} \;:\sum_{t=1}^{p_i} c_{it} z^{q_i - q_{it}} \prod_{j=1}^{n} x_j^{q_{itj}} \leq 0 \text{ for } i \in I,\, zu \geq x \geq lz,\, 1 \geq z \geq 0,\;\Big\}.$$

Proof. Note that $f_i(x/z) \;=\; \sum_{t=1}^{p_i} c_{it} z^{-q_{it}} \prod_{j=1}^{n} x_j^{q_{itj}}$. Therefore, multiplying $f_i(x/z) \;\leq\; 0$ by z^{q_i}, one obtains the expression above. Clearly, $W^c \cap \{z > 0\} = W^-$ and $W^c \cap \{z = 0\} = W^0$ ∎

3.2. **Convex hulls of convex sets.** Given a collection of bounded convex sets, it is easy to define an extended formulation to describe their convex hull using additional variables, similar to (XF). It is however, not possible to produce a description in the space of original variables. The particular case we considered in the previous section involves only two sets, one of which consists of a single point. For the sake of completeness we next summarize some related results from Ceria and Soares [10].

Ceria and Soares [10] use *perspective* functions of the functions that define the original sets to produce an extended formulation for the convex hull description. If the original sets are defined by convex functions, their perspective functions are also convex. More precisely, for $t = 1, \ldots, p$, let $G^t : \mathbb{R}^n \to \mathbb{R}^{m_t}$ be a mapping defined by convex functions and assume that the corresponding set

$$K^t = \{x \in \mathbb{R}^n \,:\, G^t(x) \leq 0\}$$

is bounded. Let $\tilde{G}^t : \mathbb{R}^{n+1} \to \mathbb{R}^{m_t}$ be the perspective mapping defined as

$$\tilde{G}^t(\lambda, x) = \begin{cases} \lambda G^t(x/\lambda) & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ \infty & \text{otherwise} \end{cases}$$

We next state a important observation from Ceria and Soares [10] that shows the use of perspective functions to obtain convex hulls of convex sets.

Lemma 9 ([10]). *Let K^t be defined as above for $t \in T = \{1, \dots, T|\}$, and let $K = \text{conv}(\cup_{t=1}^{|T|} K^t)$. Then, $x \in K$ if and only if the following nonlinear system is feasible:*

$$x = \sum_{t=1}^{|T|} x^t \,; \qquad \tilde{G}^t(\lambda_t, x^t) \leq 0, \quad \sum_{t=1}^{|T|} \lambda_t = 1, \quad \lambda_t \geq 0, \quad t \in T$$

Furthermore, all $\tilde{G}^t$ are convex mappings provided that all G^t are convex.

Put into this context, our observations in Section 3.1 specialize Lemma 9 to the case when $|T| = 2$ and one of the sets contain a single point. In this special case Corollary 1 and Lemma 8 show that a description of the convex hull in the original space can be obtained easily.

3.3. **Perspective Cuts.** Building on the work of Ceria and Soares [10], Frangioni and Gentile [12] introduce the class of *perspective cuts* for mixed integer programs of the form

$$\min_{(x,z)\in\mathbb{R}^n\times\mathbb{B}} \Big\{ f(x) + cz \mid Ax \leq bz \Big\},$$

where (i) $X = \{x \mid Ax \leq b\}$ is bounded (also implying $\{x \mid Ax \leq 0\} = \{0\}$), (ii) $f(x)$ is a convex function that is finite on X, and (iii) $f(0) = 0$. Under these assumptions, they are able to show that for any $\bar{x} \in X$ and $s \in \partial f(\bar{x})$, the following (linear) inequality

$$v \geq f(\bar{x}) + c + s^T(x - \bar{x}) + (c + f(\bar{x}) - s^T\bar{x}))(z - 1) \tag{6}$$

is valid for the equivalent mixed integer program

$$\min_{(x,z,v)\in\mathbb{R}^n\times\mathbb{B}\times\mathbb{R}} \Big\{ v \mid v \geq f(x) + cz, Ax \leq bz \Big\}.$$

Frangioni and Gentile [12] derive the inequalities (6) from a first-order analysis of the convex envelope of the perspective function of $f(x)$. A similar first-order argument can be used to derive inequality (6) from the characterization of the convex hull of the union of a convex set and a point given in Section 3. First define $P^0 \stackrel{\text{def}}{=} \big\{(x, z, v) \in \mathbb{R}^{n+2} \;:\; x = 0, z = 0, v = 0\big\}$, and

$$P^1 \stackrel{\text{def}}{=} \big\{(x, z, v) \in \mathbb{R}^{n+2} \;:\; Ax \leq b, f(x) + c - v \leq 0,\; u_x \geq x \geq l_x,\; u_v \geq v \geq l_v,\; z = 1\big\}$$

where bounds on variables x and v are introduced without loss of generality. Corollary 1 states that $\text{conv}(P^0 \cup P^1)$ is the closure of

$$P^- \stackrel{\text{def}}{=} \Big\{(x, z, v) \in \mathbb{R}^{n+2} \mid Ax \leq b,\; zf(x/z) + cz - v \leq 0, u_x z \geq x \geq l_x z,\; u_v z \geq v \geq l_v z,\; 1 \geq z \geq 0\Big\}.$$

For any $\bar{z} > 0$, a first-order (outer)-approximation of the nonlinear constraint $zf(x/z) + cz - v \leq 0$ about the point $(\bar{x}, \bar{z}, \bar{v})$ gives

$$0 \geq \bar{z}f(\bar{x}/\bar{z}) + c\bar{z} - \bar{v} + \begin{bmatrix} s \\ (-1/\bar{z})\bar{x}^T s_{x/z} + f(\bar{x}/\bar{z}) + c \\ -1 \end{bmatrix}^T \begin{bmatrix} x - \bar{x} \\ z - \bar{z} \\ v - \bar{v} \end{bmatrix},$$

where $s \in \partial f(\bar{x})$ and $s_{x/z} \in \partial f(\bar{x}/\bar{z})$. Taking $\bar{z} = 1$, $\bar{v} = f(\bar{x}) + c$, and rearranging terms gives inequality (6) above.

4. Applications

In this section, two applications are described: a quadratic uncapacitated facility location problem and a network design problem with nonlinear congestion constraints. In each case, the positive impact of the perspective reformulation and the ability to model the nonlinear inequalities in the reformulations as second-order cone constraints is demonstrated.

4.1. **Separable Quadratic UFL.** The Separable Quadratic Uncapacitated Facility Location Problem (SQUFL) was introduced by Günlük et al. [14]. In the SQUFL, there is a set of customers ($N = \{1, 2, \ldots, n\}$), a set of facilities ($M = \{1, 2, \ldots, m\}$), and each customer must have its demand for a single commodity met by an open facility. There is a fixed cost c_i for opening a facility $i \in M$. Meeting the demand of customer $j \in N$ from facility $i \in M$ costs an amount proportional to the square of the quantity delivered. A mixed integer nonlinear program for the SQUFL is

$$z^* \stackrel{\text{def}}{=} \min_{(x,z) \in \mathbb{R}^{mn}_+ \times \mathbb{B}^m} \left\{ \sum_{i \in M} c_i z_i + \sum_{i \in M} \sum_{j \in N} q_{ij} x_{ij}^2 \mid x_{ij} \leq z_i \; \forall i \in M, \forall j \in N, \; \sum_{i \in M} x_{ij} = 1 \; \forall j \in N \right\}. \tag{7}$$

The variables z_i indicate if facility $i \in N$ is open, and x_{ij} is a decision variable representing the fraction of customer j's demand met from facility i. We let z_R be the optimal solution value of the relaxation of (7) in which the constraints $z_i \in \{0, 1\}$ are replaced by $z_i \in [0, 1]$.

To write SQUFL as an indicator-induced MINLP, the auxiliary variables y_{ij} $\forall i \in M, j \in N$ are introduced. The objective function is changed to the linear function

$$\min \sum_{i \in M} c_i z_i + \sum_{i \in M} \sum_{j \in N} q_{ij} y_{ij},$$

and the constraints

$$x_{ij}^2 - y_{ij} \leq 0 \quad \forall i \in M, j \in N \tag{8}$$

are added. In this reformulation, if the indicator variable $z_i = 0$, then $x_{ij} = 0 \; \forall j \in N$ and the constraints (8) become redundant, while if $z_i = 1$, the constraints (8) become active. Thus, the constraints (8) can be replaced by their perspective counterparts

$$x_{ij}^2 - z_i y_{ij} \leq 0 \qquad \forall i \in M, \forall j \in N, \tag{9}$$

and the resulting relaxation should be significantly tighter. We will let z_P denote the optimal solution value of the relaxation of the perspective reformulation.

4.1.1. *Computational Results.* To test the strength of the perspective reformulation, random instances were constructed with facilities and locations uniformly distributed in the unit square. The fixed cost of opening facility $i \in M$ was taken to be $c_i = \lfloor \mathcal{U}(1, 100) \rfloor$. If $p_i \in [0, 1]^2$ was the location of facility $i \in M$ and $r_j \in [0, 1]^2$ was the location of customer $j \in N$, then the variable cost parameter was calculated as $q_{ij} = 50 \| p_i - r_j \|$. Günlük et al. [14] constructed instances in a similar manner. For $m \in \{10, 20, 30, 20\}$ and $n \in \{30, 50, 100, 200\}$, ten instances were created and solved using the nonlinear branch-and-bound algorithm available in the open-source MINLP code `BONMIN` [7]. The instances were solved using both the original formulation (7) and the perspective reformulation. All instances were solved on a 1.8GHz AMD Opteron CPU.

Table 1 shows the results of this experiment. In the table, $\bar{z}_R$ represents the average value of the relaxation of the original formulation, $\bar{z}_P$ the average value of the relaxation of the perspective reformulation, and $\bar{z}^*$ the average value of the optimal solution found by `BONMIN`. The table also displays the number of instances (out of 10) that were solved within a time limit of 8 hours, the average number of nodes $\bar{N}$ required to solve the instances, and the average CPU time ($\bar{T}$) in seconds for both the original and perspective formulations. Clearly, reformulating the problem via the perspective reformulation has an enormous impact on the ability to solve the problem.

The results in Table 1 indicate that the CPU time required to solve one node of the branch-and-bound tree increases dramatically when the perspective formulation is applied. `BONMIN` uses the interior-point solver `Ipopt` [24] for solving relaxations that arise at nodes of the branch-and-bound tree. `Ipopt` is a solver for general nonlinear programs and is unable to exploit the special second-order cone structure of the inequalities in the perspective reformulation. Even more disturbing is the fact that since the functions

TABLE 1. Relaxation Values and Solution Times for SQUFL

					Original Formulation			Perspective Formulation		
m	n	$\bar{z}_R$	$\bar{z}_P$	$\bar{z}^*$	# Solved	$\bar{N}$	$\bar{T}$	# Solved	$\bar{N}$	$\bar{T}$
10	30	105.8	196.5	197.9	10	333	8.9	10	15	3.7
10	50	160.4	312.6	314.6	10	406	18.0	10	11	4.9
10	100	266.5	460.4	462.0	10	441	36.7	10	9	7.7
10	200	470.7	733.6	737.0	10	350	59.7	10	7	15.2
20	30	81.7	186.1	185.6	10	3452	213.7	10	37	39.9
20	50	111.6	274.8	276.2	10	5526	601.4	10	31	85.9
20	100	166.3	412.7	414.5	7	25901	12263.9	10	35	677.1
20	200	283.5	650.8	653.1	0	-	-	10	27	1925
30	30	64.1	157.8	159.4	9	17837	1822.7	10	62	192.8
30	50	82.1	241.6	243.3	1	61062	23760.2	10	56	650.3
30	100	126.0	343.4	345.6	0	-	-	10	51	4565.4
30	200	200.7	545.8	547.4	0	-	-	9	44	16858.5
40	30	58.6	146.4	147.7	7	55660	9319.6	10	71	224.3
40	50	74.1	198.7	200.0	0	-	-	10	85	3030.6
40	100	109.6	309.8	311.2	0	-	-	10	64	8420.8
40	200	161.4	478.3	-	0	-	-	0	-	-

TABLE 2. Solution Times for SOC-Perspective Reformulation of SQUFL

m	n	T	N
30	200	141.9	63
40	100	76.4	54
40	200	101.3	45
50	100	61.6	49
50	200	140.4	47

$x^2 - yz$ appearing in the perspective reformulation are not convex, `Ipopt` cannot guarantee convergence to a stationary point and its performance is highly dependent on the quality of the initial iterate provided.

To eliminate the obstacles faced by a general NLP solver, the conic formulations were solved with Mosek [17], a code specialized for problems of this type. Table 2 shows the number of nodes (N) and CPU seconds (T) required by Mosek v5.0 to solve large random instances of SQUFL formulated with the perspective reformulation wherein the nonlinear inequalities are represented in second-order-cone form. Note the order-of-magnitude improvement in solution time, which comes solely from the reduced time to solve relaxations at nodes of the branch-and-bound tree.

Table 3, taken from Table 1 of the paper of Günlük et al. [14], shows the effectiveness of three classes of cutting planes introduced there at closing the optimality gap at the root node. In the table z_R is the value of the relaxation of the original formulation, z_{GLW} is the value of the relaxation with three classes of valid inequalities added, z_P is the value of the relaxation of the perspective reformulation, and z^* is the optimal solution value. The table shows that the perspective reformulation is significantly better at closing the integrality gap than are the cutting planes of Günlük et al. [14].

The largest of the instances in Table 3 was solved to optimality by Lee [16] using `BONMIN`. The solution required '16697 CPU seconds and 45,901 nodes for the original formulation, and a 21206 CPU seconds and 29277 nodes for the formulation with additional inequalities added. The same instance was solved using Mosek v5 on the perspective reformulation wherein the nonlinear inequalities were written as second-order cone constraints. Solution of the instance required only 44 branch-and-bound nodes and *23 CPU seconds* to solve on Intel Pentium 4 CPU with a clock speed of 2.60GHz, a speedup factor of more than 700.

TABLE 3. Comparison of Relaxation Bounds for SQUFL

m	n	z_R	z_{GLW}	z_P	z^*
10	30	140.6	326.4	346.5	348.7
15	50	141.3	312.2	380.0	384.1
20	65	122.5	248.7	288.9	289.3
25	80	121.3	260.1	314.8	315.8
30	100	128.0	327.0	391.7	393.2

4.2. **Network Design with Congestion Constraints.** In this section, a model for constructing a communication network at minimum cost meeting a design specification for total queuing delay is presented. Similar models appear in the work of Boorstyn and Frank [8], Bertsekas and Gallager [5], and Borchers and Mitchell [9]. In the problem, there is a set of commodities K to be shipped over a capacitated directed network $G = (N, A)$. The capacity of arc $(i, j) \in A$ is u_{ij}, and each node $i \in N$ supplies or demands a specified amount b_i^k of commodity k. There is a fixed cost c_{ij} of opening each arc $(i, j) \in A$, and we introduce {0-1} decision variables z_{ij} to indicate whether arc $(i, j) \in A$ is opened. The quantity of commodity k routed on arc (i, j) is measured by the decision variable x_{ij}^k. A typical function to measure the total weighted congestion (or queuing delay) of a flow $f_{ij} = \sum_{k \in K} x_{ij}^k$ in the network is

$$\rho(f) \stackrel{\text{def}}{=} \sum_{(i,j) \in A} r_{ij} \frac{f_{ij}}{1 - f_{ij}/u_{ij}},$$

where $r_{ij} \geq 0$ is a user-defined importance parameter for the queuing delay that occurs on arc (i, j). We use a decision variables y_{ij} to measure the contribution of the congestion on arc (i, j) to the total congestion $\rho(f)$. The network should be designed so as to keep the total queuing delay less than a given value β, and this is to be accomplished at minimum cost. The resulting optimization model can be written as

$$\begin{aligned}
\min_{(x,y,z,f) \in \mathbb{R}_+^{|A| \times |K|} \times \mathbb{R}_+^{|A|} \times \mathbb{B}^{|A|} \times \mathbb{R}_+^{|A|}} \quad & \sum_{(i,j) \in A} c_{ij} z_{ij} \\
\text{subject to} \quad \sum_{(j,i) \in A} x_{ij}^k - \sum_{(i,j) \in A} x_{ij}^k &= b_i^k \qquad \forall i \in N, \forall k \in K \\
\sum_{k \in K} x_{ij}^k - f_{ij} &= 0 \qquad \forall (i,j) \in A \\
f_{ij} &\leq u_{ij} z_{ij} \qquad \forall (i,j) \in A \qquad (10) \\
y_{ij} &\geq \frac{r_{ij} f_{ij}}{1 - f_{ij}/u_{ij}} \qquad \forall (i,j) \in A \qquad (11) \\
\sum_{(i,j) \in A} y_{ij} &\leq \beta
\end{aligned}$$

An observation not previously made in the literature regarding this network design problem is that the congestion inequalities (11) can be written as second-order cone constraints. Multiplying both sides of the inequality by $1 - f_{ij}/u_{ij} > 0$, adding $r_{ij} f_{ij}^2$ to both sides of the inequality, and factoring the left-hand-side gives an equivalent constraint

$$(y_{ij} - r_{ij} f_{ij})(u_{ij} - f_{ij}) \geq r_{ij} f_{ij}^2. \qquad (12)$$

Because $y_{ij} \geq r_{ij} f_{ij}$ and $u_{ij} \geq f_{ij}$, (12) is precisely a constraint in rotated second-order conic form (4).

The relaxation can be strengthened by noting that if $z_{ij} = 0$, then the constraints (10) force $f_{ij} = 0$, and the constraints (11) are redundant for the arc (i, j). However, if $z_{ij} = 1$, then the definitional constraint (11) for the corresponding y_{ij} must hold. We can then strengthen the formulation by applying the perspective reformulation. Specifically, each constraint (11) can be replaced by its perspective counterpart:

$$z_{ij}\left[\frac{r_{ij}f_{ij}/z_{ij}}{1-f/(u_{ij}z_{ij})}-\frac{y}{z}\right]\leq 0. \tag{13}$$

The constraints (13) can also be written as second order cone constraints in a similar fashion to the non-perspective version (11). Specifically, simplifying the left-hand size of the inequality (13), adding $r_{ij}f_{ij}^2$ to both sides of the simplified inequality and factoring gives the equivalent constraints

$$(y_{ij}-r_{ij}f_{ij})(u_{ij}z_{ij}-f_{ij})\geq r_{ij}f_{ij}^2,$$

which is a rotated second-order cone constraint since $y_{ij}\geq r_{ij}f_{ij}$ and $u_{ij}z_{ij}\geq f_{ij}$. The fact that the inequalities in the perspective reformulation of (11) are SOC-representable is no surprise. In fact, Ben-Tal and Nemirovski [4] (Page 96, Proposition 3.3.2) show that the perspective transformation of a function whose epigraph is a SOC-representable set is nearly always SOC-representable.

4.2.1. *Computational Results.* To assess the strength of the perspective reformulation of this nonlinear network design problem, three test instances were created. The first instance was the `atlanta` network from SNDLIB [19]. The second and third instances were generated randomly. MPS files for all of the instances are available on request from the authors.

Each of the instances in the test suite was solved using Mosek v5.0 using both the original and perspective formulations of the problem on an Intel Pentium 4 CPU with a clock speed of 2.60GHz. A time limit of one CPU hour was imposed on each run. Table 4 shows the sizes of each instance in the test suite, as well as various characteristics of the solution. z_{root} is the value of the relaxation of the root node of the branch-and-bound tree, (z_L, z_U) are the best lower and upper bounds found in one hour of CPU time, # Nodes is the number of nodes in the enumeration tree, and T is the CPU seconds on an Intel Pentium 4 CPU with a clock speed of 2.60GHz.

TABLE 4. Impact of Perspective Reformulation on Network Design Instances

				Original Form.				Perspective Form.			
Instance	$\|N\|$	$\|K\|$	$\|A\|$	z_{root}	(z_L, z_U)	# Nodes	T	z_{root}	(z_L, z_U)	# Nodes	T
ATL	15	15	22	40.7	(55.4,55.4)	752	116.9	48.3	(55.4,55.4)	464	66.8
R1	20	20	44	37.7	(135.9,172.2)	2488	3600	78.8	(147.5,158.4)	5781	3600
R2	30	30	108	46.8	(140.8,326.9)	253	3600	59.9	(201.5,∞)	394	3600

For the network design problems, the perspective formulation is always quite useful for improving the lower bounds, and in two of the cases, this translates into improved performance. For the instance R2, Mosek was unable to find a feasible solution to the instance when reformulated via the perspective transformation.

5. CONCLUSIONS

In this work we derive an explicit characterization of the convex hull of the union of a point and a bounded convex set defined by analytic functions. This characterization can be used to produce strong "perspective" reformulations of many practical mixed integer nonlinear programs. We also show that in many cases, the nonlinear inequalities in the perspective reformulation can be cast as second-order cone constraints, a transformation that greatly improves an instance's solvability. Computational results on two practical applications show the power of the proposed techniques—in one case solving instances multiple orders of magnitude faster than reported in the literature. Continuing work has two primary thrusts: (1) Automatic detection of structures to which the perspective transformation can be applied; and (2) Studying additional simple structures occurring in practical MINLPs in the hope of deriving strong relaxations.

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