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FULL LENGTH PAPER

Valid inequalities for separable concave constraints with indicator variables

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Abstract We study valid inequalities for optimization models that contain both binary indicator variables and separable concave constraints. These models reduce to a mixed-integer linear program (MILP) when the concave constraints are ignored, or to a nonconvex global optimization problem when the binary restrictions are ignored. In algorithms designed to solve these problems to global optimality, cutting planes to strengthen the relaxation are traditionally obtained using valid inequalities for the MILP only. We propose a technique to obtain valid inequalities that are based on both the MILP constraints and the concave constraints. We begin by characterizing the convex hull of a four-dimensional set consisting of a single binary indicator variable, a single concave constraint, and two linear inequalities. Using this analysis, we demonstrate how valid inequalities for the single node flow set and for the lot-sizing polyhedron can be "tilted" to give valid inequalities that also account for separable concave functions of the arc flows. We present computational results demonstrating the utility of the new inequalities for nonlinear transportation problems and for lot-sizing problems with concave costs. To our knowledge, this is one of the first works that simultaneously convexifies both nonconvex functions and binary variables to strengthen the relaxations of practical mixed-integer nonlinear programs.

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1 Introduction

Cutting planes are a crucial ingredient in modern solvers for mixed-integer linear programs (MILP). The cutting planes are typically derived through a mathematical analysis of structured sets that are relaxations of the feasible region of MILP. In recent years, there has been significant research effort applying the same paradigm to mixed-integer nonlinear programs (MINLP), whose feasible region is defined by constraint functions that are nonlinear. Most of the work in MINLP has been on the analysis of sets where the nonlinear functions are convex (see, e.g., [4,6] and references therein). There is relatively less research studying the structure of specific mixed-integer nonlinear sets where the nonlinear functions are nonconvex [17,23]. Our work adds to the body of knowledge on structured, mixed-integer nonlinear sets whose constraint functions are not convex.

We study a mixed-integer nonlinear set composed of a base polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$, a *variable bound set* composed of a collection of indicator variables

$$\mathcal{Z} := \{(x, z) \in \mathbb{R}^n \times \mathbb{B}^n : \ell_i z_i \leq x_i \leq u_i z_i \text{ for } i \in [n]\},$$

and the component-wise epigraphs of n univariate concave functions $\{f_1, \ldots, f_n\}$, $f_i : \mathbb{R} \to \mathbb{R}$,

$$\mathcal{T} := \{(x, t) \in \mathbb{R}^{2n} : t_i \geq f_i(x_i), 0 \leq x_i \leq u_i \text{ for } i \in [n]\}.$$

By studying an analogous low-dimensional set, we give a methodology for producing valid inequalities for the set

$$\mathcal{X} := \{(x, z, t) \in \mathbb{R}^n \times \mathbb{B}^n \times \mathbb{R}^n : x \in \mathcal{P}, (x, z) \in \mathcal{Z}, (x, t) \in \mathcal{T}\} \tag{1}$$

for relatively general variants of the base polytope $\mathcal{P}$.

Throughout the paper, we use the standard notation $[n] = \{1, 2, \ldots, n\}$. For each $i \in [n]$, we assume without loss of generality that $f_i(0) = 0, \ell_i < u_i$, and for simplicity of presentation we assume that $\ell_i \geq 0$. We frequently abuse notation and perform set intersection between sets with different domains. For $\mathcal{A} \subseteq \{(a, b) \in A \times B\}$ and $\mathcal{B} \subseteq \{(b, c) \in B \times C\}$, we let $\mathcal{A} \cap \mathcal{B} := \{(a, b, c) \in A \times B \times C : (a, b) \in \mathcal{A} \text{ and } (b, c) \in \mathcal{B}\}$. With this abuse of notation, we can say that $\mathcal{X} = \mathcal{P} \cap \mathcal{Z} \cap \mathcal{T}$. Using the facts that $\text{conv}(\mathcal{P} \cap \mathcal{Z})$ is a polyhedron, and $\mathcal{T}$ is the Cartesian product of the epigraphs of univariate *concave* functions, it is easy to establish that $\text{conv}(\mathcal{X})$ is a polyhedron. See e.g., Theorem I.1 in [16].

Relaxations of the set $\mathcal{X}$ appear as substructures in many important optimization problems. For example, in Sect. 4, we focus on the case where $\mathcal{P}$ is a single flow constraint, $P^{\text{SNFS}} := \left\{x \in \mathbb{R}^n_+ : \sum_{i \in N^+} x_i - \sum_{i \in N^-} x_i \leq d\right\}$, so that the set $P^{\text{SNFS}} \cap \mathcal{Z}$ is the well-studied *single node flow set*, which arises naturally in many important practical applications and for which many classes of strong valid inequalities are known

[11,22]. When $\mathcal{P}$ is the network-flow polytope, the set $\mathcal{P} \cap \mathcal{T}$ is the feasible region of the *minimum concave-cost network flow problem* (MCNFP) [13]. The MCNFP arises in many application areas, including communication network design, facility location, and VLSI design, where the concave functions typically model economies of scale. In lot-sizing problems, $\mathcal{P}$ is a network-flow polytope, and the set $\mathcal{P} \cap \mathcal{Z}$ is used to model a production plan that has both fixed and variable cost components. In Sect. 5, we demonstrate how to apply our analysis to derive new classes of valid inequalities for lot-sizing problems that have a concave variable-cost structure, so that $\mathcal{X} = \mathcal{P} \cap \mathcal{Z} \cap \mathcal{T}$ is the relevant mathematical structure to study.

The set $\mathcal{X}$ also occurs as a relaxation of formulations of engineering design problems that involve a non-linear relationship between input and output variables. For example, in water-network design problems, the potential loss t_a across an arc $a \in A$ in the pipe network can be modeled as a nonlinear function of the water flow x_a, $t_a = f_a(x_a) := \alpha_a \operatorname{sign}(x_a)x_a^2$ [8]. Similar nonlinear, non-convex relationships exist in modeling the pressure loss along pipes in a gas network [19,25] or to model the efficiency of hydro-electric power generation [7]. In each of these cases, the set $\mathcal{T}$ is used to model the nonlinearity, the base polytope $\mathcal{P}$ captures important physical properties such as flow balance, and the indicator structure in the set $\mathcal{Z}$ may be used to model discrete pipe sizes, the uni-directionality of flows, or discrete start-up-behavior of components of the engineered system under study. In this case, even though the nonlinear functions are not necessarily concave, to build a valid relaxation, algorithms must generate relaxations for both the convex and concave parts of the function [9,20,24]. Thus, valid inequalities for the set $\mathcal{X}$ could be an important building block in algorithms for solving engineering design problems.

The paper has six subsequent sections. In Sect. 2, we show that $\operatorname{conv}(\mathcal{Z} \cap \mathcal{T})$ is a simple polyhedral set consisting of a strengthened version of secant inequalities from $\mathcal{T}$. Thus, we argue that in order to get stronger relaxations of $\mathcal{X}$, we must simultaneously consider each of the sets whose intersection forms $\mathcal{X} = \mathcal{P} \cap \mathcal{Z} \cap \mathcal{T}$. In Sect. 3, we build a simple, low-dimensional set derived from valid inequalities for $\mathcal{P} \cap \mathcal{Z}$, and we describe how to use this set to construct a valid inequality for $\mathcal{X}$. In Sect. 4, we demonstrate how to apply the methodology from Sect. 3 when the set $\mathcal{P} \cap \mathcal{Z}$ is the single node flow set. We derive a new class of strong valid inequalities called *Tilted Simple Generalized Flow Cover Inequalities* (TSGFCI) for $\mathcal{X}$. In Sect. 5, we study the lot sizing problem where economies of scale are modeled with concave variable costs. Extensions of known inequalities from lot sizing, which we call the *tilted* (ℓ, S) *inequalities* (TLSI), are derived. Section 6 contains two computational studies demonstrating how using both the TSGFCI and TLSI inequalities can lead to significant speedups to state-of-the-art global optimization software. We make concluding remarks in Sect. 7.

A preliminary version of this work appeared as the extended abstract [18]. The important extensions made in the current paper include the following.

– We generalized the low-dimensional set studied in Sect. 3, which allowed us to strengthen an earlier result on TSGFCI to include both incoming and outgoing arcs from the single node flow set.

- Section 5, showing the application of our methodology on lot-sizing problems is completely new, as is the corresponding computational study in Sect. 6.2.
- The proofs of all technical results, which were excluded due to space considerations in [18] are included in this work.
- The exposition has been improved by adding more examples.

2 Motivation: the set $\mathcal{Z} \cap \mathcal{T}$

The standard methodology used to solve optimization problems involving non-convex structures such as $\mathcal{X}$ to global optimality is to create a convex relaxation of $\mathcal{X}$ and then to refine the relaxation over the feasible region via a branch-and-bound approach. The most natural way to create a convex relaxation of $\mathcal{X}$, employed by state-of-the-art software such as BARON [24], ANTIGONE [20], and SCIP [5], is to relax the integrality requirements on binary variables in $\mathcal{Z}$,

$$\mathcal{R}(\mathcal{Z}) := \{(x, z) \in \mathbb{R}^n \times [0, 1]^n : \ell_i z_i \le x_i \le u_i z_i \text{ for } i \in [n]\},$$

and to underestimate the concave functions $f_i(\cdot)$ using the secant intersecting the graph of the function at the endpoints of its domain,

$$\mathcal{S}(\mathcal{T}) := \left\{(x, t) \in \mathbb{R}^{2n} : t_i \ge f_i(\ell_i) + \frac{f_i(u_i) - f_i(\ell_i)}{u_i - \ell_i}(x_i - \ell_i) \text{ for } i \in [n]\right\}.$$

The polyhedron $\mathcal{R}(\mathcal{X}) := \mathcal{P} \cap \mathcal{R}(\mathcal{Z}) \cap \mathcal{S}(\mathcal{T})$ is a relaxation that can be employed within a branch-and-bound approach to optimize over $\mathcal{X}$.

The constraints in the set $\mathcal{Z}$ enforce the logical conditions that if the binary variable $z_i = 0$, then the associated variable $x_i = 0$ as well. Using this fact, one can strengthen the relaxation $\mathcal{R}(\mathcal{X})$ using the set of *strengthened secant inequalities*

$$\bar{\mathcal{S}}(\mathcal{T}, \mathcal{Z}) := \left\{(x, z, t) \in \mathbb{R}^{3n} : t_i \ge f_i(\ell_i) z_i + \frac{f_i(u_i) - f_i(\ell_i)}{u_i - \ell_i}(x_i - \ell_i z_i) \text{ for } i \in [n]\right\}.$$

The set $\bar{\mathcal{S}}(\mathcal{T}, \mathcal{Z})$ forms the basis of the strongest possible convex relaxation of the $\mathcal{X}$ when the constraints in $\mathcal{P}$ are ignored.

Proposition 1 *conv*$(\mathcal{Z} \cap \mathcal{T}) = \mathcal{R}(\mathcal{Z}) \cap \bar{\mathcal{S}}(\mathcal{T}, \mathcal{Z})$.

Proof Since the triples of variables (x_j, z_j, t_j) are all independent in $\mathcal{Z} \cap \mathcal{T}$, it suffices to prove the result for the case $n = 1$. We let Z and T denote the $n = 1$ instances of the sets $\mathcal{Z}$ and $\mathcal{T}$, respectively. For a set S that has a single binary variable, let S^j denote the set with the binary variable fixed to $j \in \{0, 1\}$. Using this disjunction, we have that

$$\text{conv}(Z \cap T) = \text{conv}\Big((Z \cap \text{conv}(T))^0 \cup (Z \cap \text{conv}(T))^1\Big).$$

The fact that

$$\operatorname{conv}\Big((Z \cap \operatorname{conv}(T))^0 \cup (Z \cap \operatorname{conv}(T))^1\Big) = \mathcal{R}(Z) \cap \bar{\mathcal{S}}(Z, T)$$

can be established using standard results on the convex hull of the union of polyhedra [2]. □

Proposition 1 implies that the strengthened secant inequalities yield the strongest relaxation we can obtain of $\mathcal{X}$ if we ignore the interaction between $\mathcal{P}$ and $\mathcal{Z} \cap \mathcal{T}$. Thus, we next investigate a methodology that can simultaneously consider portions of all components of the structure of $\mathcal{P} \cap \mathcal{Z} \cap \mathcal{T}$.

3 A low-dimensional mixed-integer nonlinear set

Our goal is to derive valid inequalities for the set $\mathcal{P} \cap \mathcal{Z} \cap \mathcal{T}$ from valid inequalities for the set $\mathcal{P} \cap \mathcal{Z}$. To that end, we define the following low-dimensional mixed-integer linear sets

$$S_{\geq} := \Big\{(s, x, z) \in \mathbb{R}^2 \times \mathbb{B} \ : \ s + a_i x + b_i z \geq \gamma, \ i = 1, 2, \ \ell z \leq x \leq uz\Big\}$$

and

$$S_{\leq} := \Big\{(s, x, z) \in \mathbb{R}^2 \times \mathbb{B} \ : \ s + a_i x + b_i z \leq \gamma, \ i = 1, 2, \ \ell z \leq x \leq uz\Big\}.$$

We assume that the two linear inequalities in the definitions of $S_{\geq}$ and $S_{\leq}$ intersect in the $z = 1$ plane at a point whose x coordinate is m, where $\ell < m < u$. Thus, we make the assumption that the coefficients $a_i, b_i, i = 1, 2$, satisfy

$$\ell < m := \frac{b_2 - b_1}{a_1 - a_2} < u. \tag{2}$$

This assumption implies that $a_1\ell + b_1 < a_2\ell + b_2$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function with $f(0) = 0$. We assume that

$$f(m) > f(\ell) + \left(\frac{f(u) - f(\ell)}{u - \ell}\right)(m - \ell) \tag{3}$$

since otherwise f is affine in the range $[\ell, u]$. We are interested in studying valid inequalities for the following mixed-integer nonlinear sets:

$$ST_{\geq} := \{(s, x, z, t) \in \mathbb{R}^2 \times \mathbb{B} \times \mathbb{R} \ : \ (s, x, z) \in S_{\geq}, \ t \geq f(x)\}$$
$$ST_{\leq} := \{(s, x, z, t) \in \mathbb{R}^2 \times \mathbb{B} \times \mathbb{R} \ : \ (s, x, z) \in S_{\leq}, \ t \geq f(x)\}.$$

The analysis of the sets $ST_{\geq}$ and $ST_{\leq}$ is nearly identical, so we focus our analysis on $ST_{\geq}$ and then just present the main results for $ST_{\leq}$.

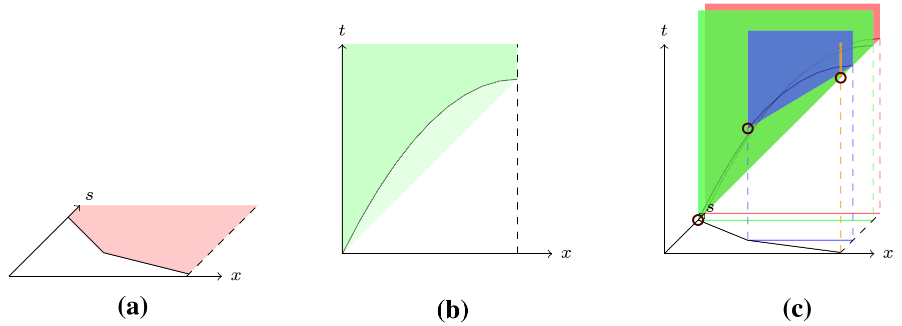

Fig. 1 Visualizing the set $ST^1_{\geq}$. **a** The projection of $ST^1_{\geq}$ onto (s, x). **b** A slice of the set $ST^1_{\geq}$ for a fixed value of s, along with the secant inequality. **c** Slices of the set $ST^1_{\geq}$ along four s values. The three extreme points are circled

We begin by analyzing the extreme points of $\mathrm{conv}(ST_{\geq})$. All extreme points of $\mathrm{conv}(ST_{\geq})$ have $z = 0$ or $z = 1$, so we first consider the set that is obtained when $z = 1$,

$$ST^1_{\geq} := \left\{(s, x, t) \in \mathbb{R}^3 \;:\; s + a_i x \geq \gamma - b_i,\; i = 1, 2,\; \ell \leq x \leq u,\; t \geq f(x)\right\}.$$

Figure 1 helps visualize the set $ST^1_{\geq}$. It follows from concavity of f that the extreme points of $\mathrm{conv}(ST^1_{\geq})$ are the points

$$\begin{aligned}
&(\gamma - (a_1\ell + b_1), \ell, f(\ell)),\\
&(\gamma - (a_2 m + b_2), m, f(m)), \text{ and}\\
&(\gamma - (a_2 u + b_2), u, f(u)).
\end{aligned}$$

(The s component of the second extreme point is also equal to $\gamma - (a_1 m + b_1)$.) The hyperplane defined by these three points defines a valid inequality for the set $\mathrm{conv}(ST^1_{\geq})$.

This discussion is formalized and extended to the set $ST_{\geq}$ in the following proposition.

Proposition 2 *The extreme rays of* $\mathrm{conv}(ST_{\geq})$ *are given by* $(1, 0, 0, 0)$ *and* $(0, 0, 0, 1)$*, and the extreme points of* $\mathrm{conv}(ST_{\geq})$ *are given by the points:*

$$\begin{aligned}
v^1 &= (\,\gamma, & 0,\; 0,\; 0),\\
v^2 &= (\,\gamma - (a_1\ell + b_1), & \ell,\; 1,\; f(\ell)),\\
v^3 &= (\,\gamma - (a_2 m + b_2), & m,\; 1,\; f(m)),\\
v^4 &= (\,\gamma - (a_2 u + b_2), & u,\; 1,\; f(u)).
\end{aligned}$$

Proof It is necessary to analyze the extreme rays and extreme points in the cases $z = 0$ and $z = 1$. It is easily seen that $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$ are the extreme rays of $\mathrm{conv}(ST_{\geq})$ in both cases. When $z = 0$, the set reduces to the constraints $s \geq \gamma, x = 0, t \geq 0$. The sole extreme point of this set is v^1. When $z = 1$, the set reduces to the set $ST^1_{\geq}$. If an arbitrary linear function $\lambda^s s + \lambda^x x + \lambda^t t$ is minimized over this set, then the problem is unbounded if $\lambda^t < 0$; otherwise, there is an optimal solution with $t = f(x)$. Thus, the problem reduces to minimizing the concave function $\lambda^s s + \lambda^x x + \lambda^t f(x)$ over the set

$$\text{proj}_{s,x}(ST^1_{\geq}) := \{(s, x) : s + a_i x \geq \gamma - b_i, i = 1, 2, \ell \leq x \leq u\}.$$

Extreme points v^2, v^3, and v^4 then follow since there always exists an optimal solution of this concave minimization problem at an extreme point of $\text{proj}_{s,x}(ST^1_{\geq})$. □

By polarity, (see e.g. Theorem 5.2 of [21]), we obtain the following characterization of valid inequalities for $ST_{\geq}$.

Corollary 1 *An inequality*

$$\lambda^s s + \lambda^x x + \lambda^z z + \lambda^t t \geq \lambda^0 \tag{4}$$

is valid for conv$(ST_{\geq})$ *if and only if* $(\lambda^s, \lambda^x, \lambda^z, \lambda^t, \lambda^0) \in C_{\geq}$, *where* $C_{\geq}$ *is the polyhedral cone*

$$C_{\geq} := \{\lambda \in \mathbb{R}^5 : \lambda^s \geq 0, \lambda^t \geq 0, v^k \tilde{\lambda} \geq \lambda^0,\ k = 1, \ldots, 4\},$$

where $\tilde{\lambda} := (\lambda^s, \lambda^x, \lambda^z, \lambda^t)$ *and* v^k *are defined in Proposition 2.*

Furthermore, (4) is a facet-defining inequality for conv$(ST_{\geq})$ *if and only if* λ *is an extreme ray of* $C_{\geq}$.

We are interested in valid inequalities for $ST_{\geq}$ that have $\lambda^t > 0$ and $\lambda^s > 0$. Under this condition, the characterization of valid inequalities for $ST_{\geq}$ in Corollary 1 reduces to a system of four inequalities, one for each of the points v^k, $k = 1, 2, 3, 4$, in five unknowns. Thus, the only extreme ray of that system must satisfy all four inequalities as an equality. Adding the normalization condition that $\lambda^s = 1$ and then observing that $v^1\lambda = \lambda^0$ implies that $\lambda^0 = \gamma\lambda^s = \gamma$, we obtain the following reduced system of equations:

$$\begin{pmatrix} \ell & 1 & f(\ell) \\ m & 1 & f(m) \\ u & 1 & f(u) \end{pmatrix} \begin{pmatrix} \lambda^x \\ \lambda^z \\ \lambda^t \end{pmatrix} = \begin{pmatrix} a_1\ell + b_1 \\ a_2 m + b_2 \\ a_2 u + b_2 \end{pmatrix}. \tag{5}$$

The assumption (3) together with $\ell < m < u$ implies that the system (5) has a unique solution, which we denote by $(\bar{\lambda}^x, \bar{\lambda}^z, \bar{\lambda}^t)$. We thus obtain the following valid inequality.

Theorem 1 *Let* $(\bar{\lambda}^x, \bar{\lambda}^z, \bar{\lambda}^t)$ *be the solution to (5). The inequality*

$$s + \bar{\lambda}^x x + \bar{\lambda}^z z + \bar{\lambda}^t t \geq \gamma \tag{6}$$

is a valid and facet-defining inequality for conv$(ST_{\geq})$.

Proof It remains to prove that $\bar{\lambda}^t > 0$. Consider the function $g(x) := \bar{\lambda}^x x + \bar{\lambda}^z + \bar{\lambda}^t f(x)$. Note that as f is concave and satisfies (3), $g(x)$ is either concave or convex depending on the sign of $\bar{\lambda}^t$, and is affine if and only if $\bar{\lambda}^t = 0$. We can rewrite (5) as the three equalities $g(\ell) = a_1\ell + b_1$, $g(m) = a_2 m + b_2$, and $g(u) = a_2 u + b_2$. Let $\delta := (u - m)/(u - \ell)$ and observe that

$$\delta g(\ell) + (1-\delta)g(u) < \delta(a_2\ell + b_2) + (1-\delta)(a_2 u + b_2) = a_2 m + b_2 = g(m)$$

where the first inequality follows from the assumption $a_1\ell + b_1 < a_2\ell + b_2$ and the second equation follows because $\delta\ell + (1-\delta)u = m$. It follows that g is concave and is not affine, and hence $\bar{\lambda}^t > 0$. □

The facet-defining inequalities for $\operatorname{conv}(ST_{\geq})$ that have $\lambda^s = 0$ are characterized in Proposition 1. The facet-defining inequalities for $ST_{\geq}$ that have $\lambda^t = 0$ are those that define $\operatorname{conv}(S_{\geq})$. Combining these valid inequalities with the sole facet-defining inequality for $\operatorname{conv}(ST_{\geq})$ from Theorem 1 yields a complete characterization of $\operatorname{conv}(ST_{\geq})$.

Theorem 2 *$conv(ST_{\geq})$ is described by the set of (s, x, z, t) for which $(s, x, z) \in conv(S_{\geq})$, and which satisfy (6) and the strengthened secant inequality*

$$t \geq f(\ell)z + \left(\frac{f(u) - f(\ell)}{u - \ell}\right)(x - \ell z). \tag{7}$$

Nearly identical arguments yield the following analogous result for $\operatorname{conv} ST_{\leq}$.

Theorem 3 *Let $(\bar{\lambda}^x, \bar{\lambda}^z, \bar{\lambda}^t)$ be the unique solution to the system of equations*

$$\begin{pmatrix} \ell & 1 & f(\ell) \\ m & 1 & f(m) \\ u & 1 & f(u) \end{pmatrix}\begin{pmatrix} \lambda^x \\ \lambda^z \\ \lambda^t \end{pmatrix} = \begin{pmatrix} a_2\ell + b_2 \\ a_2 m + b_2 \\ a_1 u + b_1 \end{pmatrix}. \tag{8}$$

Then, the inequality

$$s + \bar{\lambda}^x x + \bar{\lambda}^z z + \bar{\lambda}^t t \leq \gamma \tag{9}$$

is valid and facet-defining for $convST_{\leq}$, and $convST_{\leq}$ is described by the set of (s, x, z, t) with $(s, x, z) \in conv(S_{\leq})$ that satisfy (7) and (9).

Example: Let $f(x) = -x^2$ and consider the set

$$\widehat{S_{\leq}} := \left\{(s, x, z) \in \mathbb{R}^2 \times \mathbb{B} \;:\; s + x - 7z \leq 1,\; s \leq 1,\; 0 \leq x \leq 8z\right\}.$$

The linear system (8) is

$$\begin{pmatrix} 0 & 1 & 0 \\ 7 & 1 & -49 \\ 8 & 1 & -64 \end{pmatrix}\begin{pmatrix} \lambda^x \\ \lambda^z \\ \lambda^t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and has the solution $(\lambda^x, \lambda^z, \lambda^t) = (-\frac{7}{8}, 0, -\frac{1}{8})$, so Theorem 3 states that the inequality

$$s - \frac{1}{8}(t + 7x) \leq 1$$

is valid and facet-defining for $\operatorname{conv}(\widehat{S_{\leq}} \cap \{(x, t) \ : \ t \geq f(x)\})$.

4 Application to single node flow set

The *Single Node Flow Set* X^{SNFS} is

$$X^{\mathrm{SNFS}} := \Big\{(x, z) \in \mathbb{R}^n \times \mathbb{B}^n : \sum_{i \in N^+} x_i - \sum_{i \in N^-} x_i \leq d, \\ 0 \leq x_i \leq u_i z_i \text{ for } i \in N\Big\}. \tag{10}$$

N^+ and N^- denote the set of indices corresponding to the inflow and outflow arcs respectively, $N = N^+ \cup N^-$, and $n = |N|$.

We now define a variant of the Single Node Flow Set that incorporates concave functions of the flow variables. The *Concave Single Node Flow Set* is

$$X_f^{\mathrm{CSNFS}} := \Big\{(x, z, t) \in \mathbb{R}^n \times \mathbb{B}^n \times \mathbb{R}^n : (x, z) \in X^{\mathrm{SNFS}}, t_i \geq f_i(x_i) \text{ for } i \in N\Big\}$$

where $f_i : \mathbb{R} \to \mathbb{R}$ are concave functions with $f_i(0) = 0$.

4.1 Valid inequalities for X_f^{CSNFS}

Valid and facet-defining inequalities for $\operatorname{conv}(X^{\mathrm{SNFS}})$ are still valid and facet-defining for $\operatorname{conv}(X_f^{\mathrm{CSNFS}})$. We can use the theory developed in Sect. 3 to derive additional valid inequalities for X_f^{CSNFS} based on valid inequalities for X^{SNFS}. More precisely, whenever we have two valid inequalities for X_f^{CSNFS} (or X^{SNFS}) that can be written as $s + a_1 x_k + b_1 z_k \leq \gamma$ and $s + a_2 x_k + b_2 z_k \leq \gamma$ where

$$s = \sum_{i \in N \setminus \{k\}} (\pi_i^x x_i + \pi_i^z z_i + \pi_i^t t_i), \tag{11}$$

we can directly apply Theorem 3 to obtain a new valid inequality of X_f^{CSNFS}.

We begin by assuming we have a valid inequality of the form

$$\sum_{i \in M^+ \setminus F} (\alpha_i x_i + \beta_i z_i) + \sum_{i \in F} (\lambda_i^x x_i + \lambda_i^z z_i + \lambda_i^t t_i) + \sum_{i \in N \setminus M^+} (\pi_i^x x_i + \pi_i^z z_i) \leq \gamma \tag{12}$$

where $F \subset M^+ \subseteq N^+$, and for all $i \in M^+ \setminus F$ we have $\alpha_i > 0$ and $\beta_i < 0$.

To apply Theorem 3, we choose $k \in M^+ \backslash F$, and write the inequality (12) as

$$s + \alpha_k x_k + \beta_k z_k \leq \gamma$$

where

$$s = \sum_{i \in M^+ \backslash (F \cup \{k\})} (\alpha_i x_i + \beta_i z_i) + \sum_{i \in F} (\lambda_i^x x_i + \lambda_i^z z_i + \lambda_i^t t_i) + \sum_{i \in N \backslash M^+} (\pi_i^x x_i + \pi_i^z z_i). \tag{13}$$

We now establish that $s \leq \gamma$ is also a valid inequality.

Lemma 1 *Assume* (12) *is a valid inequality for* X_f^{CSNFS}. *Then* $s \leq \gamma$ *is also a valid inequality for* X_f^{CSNFS}.

Proof Let $(x, z, t) \in X_f^{\text{CSNFS}}$, and s be as defined in (13). Define the vector (x', z', t') by $(x_i', z_i', t_i') = (x_i, z_i, t_i)$ for $i \neq k$, and $(x_k', z_k', t_k') = (0, 0, 0)$. Since $\sum_{i \in N^+} x_i' - \sum_{i \in N^-} x_i' \leq \sum_{i \in N^+} x_i - \sum_{i \in N^-} x_i$, it follows from $(x, z) \in X^{\text{SNFS}}$ that also $(x', z') \in X^{\text{SNFS}}$, and thus also $(x', z', t') \in X_f^{\text{CSNFS}}$, since $f_k(0) = 0$. But, since $k \in M^+ \backslash F$, s as defined in (13) does not depend on t_k, z_k or x_k. Thus, substituting (x', z', t') into (13) to obtain s' yields $s' = s$. Finally, as $(x', z', t') \in X_f^{\text{CSNFS}}$ and (12) is valid for X_f^{CSNFS}, it holds that $s = s' + \alpha_k x_k' + \beta_k z_k' \leq \gamma$. □

Now, we assume we know the following inequality is valid for X^{SNFS}:

$$\sum_{i \in M^+} (\alpha_i x_i + \beta_i z_i) + \sum_{i \in N \backslash M^+} (\pi_i^x x_i + \pi_i^z z_i) \leq \gamma. \tag{14}$$

By repeatedly applying Theorem 3 and Lemma 1, we derive a family of valid inequalities for X_f^{CSNFS}.

Theorem 4 *Assume* (14) *is a valid inequality for* X^{SNFS} *with* $\alpha_i > 0$, $\beta_i < 0$, *and* $\alpha_i u_i + \beta_i > 0$ *for* $i \in M^+ \subseteq N^+$. *Let* $F \subseteq M^+$, *and for* $i \in F$ *let* $(\bar{\lambda}_i^x, \bar{\lambda}_i^z, \bar{\lambda}_i^t)$ *be the solution to* (8) *with* $(a_1, b_1, a_2, b_2, \ell, u) = (\alpha_i, \beta_i, 0, 0, 0, u_i)$. *Then, the following* tilted *inequality is valid for* X_f^{CSNFS}:

$$\sum_{i \in M^+ \backslash F} (\alpha_i x_i + \beta_i z_i) + \sum_{i \in F} (\bar{\lambda}_i^x x_i + \bar{\lambda}_i^t t_i) + \sum_{i \in N \backslash M^+} (\pi_i^x x_i + \pi_i^z z_i) \leq \gamma. \tag{15}$$

Proof Starting with (14) we choose $k \in M^+$ and apply Lemma 1 and Theorem 3 to obtain a valid inequality of the form (15) in which $F = \{k\}$. Note that since a_2 and b_2 are zero, the corresponding $\bar{\lambda}^z$ is also zero. Proceeding inductively, given any inequality of the form (15), we can again choose $k' \in M^+ \backslash F$ and apply the same procedure, as long as $F \subset M^+$. □

Since each inequality (15) is derived from a single base inequality (14), we refer to the process of deriving this inequalities as *tilting*.

Given a solution $(\hat{x}, \hat{z}, \hat{t})$ to the relaxation and a "base" valid inequality (14), the most violated tilted inequality (15) is obtained by setting

$$F^* := \left\{ i \in M^+ : \alpha_i \hat{x}_i + \beta_i \hat{z}_i < \bar{\lambda}_i^x \hat{x}_i + \bar{\lambda}_i^t \hat{t}_i \right\}. \tag{16}$$

We leave it as an open question to determine general conditions under which the tilting procedure yields facet-defining inequalities. However, in the next section we provide such conditions when the base inequality comes from a particular class of valid inequalities for X^{SNFS}.

4.2 Tilting flow cover inequalities

An important class of valid inequalities for the X^{SNFS}, are known as *flow cover inequalities* (FCI).

A *generalized flow cover* is defined by sets (C^+, C^-), where $C^+ \subseteq N^+$, $C^- \subseteq N^-$ and $\sum_{i \in C^+} u_i - \sum_{i \in C^-} u_i = d + \mu$, $\mu > 0$.

There are many variants of flow cover inequalities, including FCI with inflows-only [22], simple generalized and extended generalized FCI [21,26], and lifted versions of FCI and simple generalized FCI [11,12]. As an illustration of our results, we focus on the *Simple Generalized Flow Cover Inequality* (SGFCI), which can be written as

$$\sum_{i \in C^+} \left(x_i - (u_i - \mu)^+ z_i \right) - \sum_{i \in L^-} \min(u_i, \mu) z_i - \sum_{i \in N^- \setminus (C^- \cup L^-)} x_i \le d(C^+, C^-) \tag{17}$$

where (C^+, C^-) is a generalized flow cover, $L^- \subseteq N^- \setminus C^-$, and $d(C^+, C^-) := d + \sum_{i \in C^-} u_i - \sum_{i \in C^+} (u_i - \mu)^+$. Van Roy and Wolsey [26] provide sufficient conditions for the SGFCI to be facet-defining.

Tilting based on Theorem 4. If we let $M^+ = \{i \in C^+ : u_i > \mu\}$, then the SGFCI takes the form of (14) with $\gamma = d(C^+, C^-)$. Choose $F \subseteq M^+$, and for $i \in F$, let $(\bar{\lambda}_i^x, 0, \bar{\lambda}_i^t)$ be the solution to (8) with $(a_1, b_1, a_2, b_2, \ell, u) = (1, \mu - u_i, 0, 0, 0, u_i)$. Then, applying Theorem 4 we obtain that the inequality

$$\sum_{i \in C^+ \setminus F} \left(x_i - (u_i - \mu)^+ z_i \right) - \sum_{i \in L^-} \min(u_i, \mu) z_i - \sum_{i \in N^- \setminus (C^- \cup L^-)} x_i + \sum_{i \in F} \left(\bar{\lambda}_i^x x_i + \bar{\lambda}_i^t t_i \right) \le d(C^+, C^-) \tag{18}$$

is valid for X_f^{CSNFS}.

We next provide sufficient conditions for which inequality (18) is facet-defining for X_f^{CSNFS}, which generalizes Theorem 6 of [26]. The proof of the result is given in the Appendix.

Theorem 5 *Assume (i)* $d > 0$*, (ii)* $\max_{i \in C^+} u_i > \mu$*, (iii)* $u_i > \mu$ *for* $i \in L^-$*, (iv)* $C^- = \emptyset$*, (v)* $u_i > \mu$ *for all* $i \in F$*, and (vi)* $\sum_{i \in C^+ \setminus F} u_i > \mu$*. Then (18) is facet-defining for* X_f^{CSNFS}*.*

Assumptions (i)–(iv) are from [26]. Assumption (v) is the requirement for (18) to be a valid inequality, and assumption (vi) is a new facet-defining condition when F is nonempty.

Tilting based on different L^- *choices.* Theorem 4 only applies to indices in the inflow. We can leverage the particular structure of SGFCI and apply a different tilting process to certain indices in the outflow.

Theorem 6 *Let* $F \subseteq M^+$ *and* $G \subseteq \{i \in L^- : \mu < u_i\}$*. For* $i \in F$*, let* $(\bar{\lambda}_i^x, 0, \bar{\lambda}_i^t)$ *be the solution to (8) with* $(a_1, b_1, a_2, b_2, \ell, u) = (1, \mu - u_i, 0, 0, 0, u_i)$ *and for* $i \in G$*, let* $(\bar{\lambda}_i^x, 0, \bar{\lambda}_i^t)$ *be the solution to (8) with* $(a_1, b_1, a_2, b_2, \ell, u) = (0, -\mu, -1, 0, 0, u_i)$*. Then, the* Tilted Simple Generalized Flow Cover Inequality *(TSGFCI):*

$$\sum_{i \in C^+ \setminus F} \left(x_i - (u_i - \mu)^+ z_i\right) - \sum_{i \in L^- \setminus G} \min(u_i, \mu) z_i - \sum_{i \in N^- \setminus (C^- \cup L^-)} x_i + \sum_{i \in F \cup G} \left(\bar{\lambda}_i^x x_i + \bar{\lambda}_i^t t_i\right) \leq d(C^+, C^-) \tag{19}$$

is valid for X_f^{CSNFS}*.*

Proof The proof is by induction on the size of G. When $G = \emptyset$, (19) reduces to (18), which is valid by Theorem 4. Thus, assume for induction that (19) is valid for any $L^- \subseteq N^- \setminus C^-$, $F \subseteq M^+$, $G \subseteq \{i \in L^- : \mu < u_i\}$ with $|G| \leq c$, where $c \geq 0$. Consider now an $\bar{L}^- \subseteq N^- \setminus C^-$, $\bar{F} \subseteq M^+$, $\bar{G} \subseteq \{i \in L^- : \mu < u_i\}$ with $|\bar{G}| = c+1$. Choose $k \in \bar{G}$ and let

$$s = \sum_{i \in C^+ \setminus F} \left(x_i - (u_i - \mu)^+ z_i\right) - \sum_{i \in \bar{L}^- \setminus \bar{G}} \min(u_i, \mu) z_i - \sum_{i \in N^- \setminus (C^- \cup \bar{L}^-)} x_i + \sum_{i \in F \cup \bar{G} \setminus \{k\}} \left(\bar{\lambda}_i^x x_i + \bar{\lambda}_i^t t_i\right).$$

Then, the inequalities $s - \min(u_k, \mu) z_k \leq d(C^+, C^-)$ and $s - x_k \leq d(C^+, C^-)$ are valid by the induction hypothesis as they correspond to inequalities of the form (19) with $(L^-, G) = (\bar{L}^-, \bar{G} \setminus \{k\})$ and $(L^-, G) = (\bar{L}^- \setminus \{k\}, \bar{G} \setminus \{k\})$, respectively. We then apply Theorem 3 with $(a_1, b_1, a_2, b_2, \ell, u) = (0, -\mu, -1, 0, 0, u_k)$ and obtain $m = \mu$, which satisfies $0 < m < u_k$ by the assumption on G. The first equation in (8) implies $\bar{\lambda}_k^z = 0$, and thus Theorem 3 yields the valid inequality $s + \bar{\lambda}_k^x x_k + \bar{\lambda}_k^t t_k \leq d(C^+, C^-)$, which completes the induction. □

Example: Suppose we have inflows with capacities $u = (2, 3, 5, 8)$ with external demand $d = 10$ and no outflows. Let $f(x) = -x^2$. A valid flow cover is $\{2, 4\}$ since $u_2 + u_4 = 3 + 8 = d + 1$ and it gives us the following (facet-defining) simple flow cover inequality:

$$(x_2 + (3 - 1)(1 - z_2)) + (x_4 + (8 - 1)(1 - z_4)) \leq 10$$

which simplifies to $(x_2 - 2z_2) + (x_4 - 7z_4) \leq 1$. The tilting process gives us three more (facet-defining) inequalities:

$$(x_2 - 2z_2) - \frac{1}{8}(t_4 + 7x_4) \leq 1$$
$$-\frac{1}{3}(t_2 + 2x_1) + (x_4 - 7z_4) \leq 1$$
$$-\frac{1}{3}(t_2 + 2x_1) - \frac{1}{8}(t_4 + 7x_4) \leq 1$$

The first two inequalities given satisfy the assumptions of Theorem 5, while the third does not.

5 Application to lot-sizing

The *Lot-Sizing Set* X^{LS} is

$$X^{\mathrm{LS}} := \left\{(x, y, z) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{B}^n : x_i + y_{i-1} - y_i = d_i,\ x_i \leq u_i z_i \text{ for } i \in [n]\right\}.$$

Here, for each $i \in [n]$, d_i and u_i represent the demand and capacity in period i, respectively, x_i and y_i are decision variables representing the production amount and ending inventory in period i, respectively, and z_i is a binary indicator variable equal to one when production is positive in period i. We assume $y_0 = 0$. To model concave production costs, we introduce the *Concave Lot-Sizing Set* X^{CLS}:

$$X^{\mathrm{CLS}} := \Big\{(x, y, z, t) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{R}^n : (x, y, z) \in X^{\mathrm{LS}},$$
$$t_i \geq f_i(x_i) \text{ for } i \in [n]\Big\}$$

where $f_i : \mathbb{R} \to \mathbb{R}$ are concave functions with $f_i(0) = 0$.

For $1 \leq i \leq \ell \leq n$, let $D_{i\ell} := \sum_{j=i}^{\ell} d_j$ be the cumulative demand between periods i and ℓ. For any $\ell \in [n]$ and $S \subseteq \{1, \ldots, \ell\}$, the following (ℓ, S) *inequality* is valid for X^{LS} [3]:

$$\sum_{i \in S} x_i - \sum_{i \in S} D_{i\ell} z_i - y_\ell \leq 0. \tag{20}$$

In the case of the *uncapacitated lot-sizing problem*, where $u_i = D_{in}$ for $i \in [n]$, the (ℓ, S) inequalities, together with the inequalities defining X^{LS} are sufficient to define $\mathrm{conv}(X^{\mathrm{LS}})$ [3].

Given $\ell \in [n]$ and $S \subseteq \{1, \ldots, \ell\}$, $S \neq \emptyset$, choose $k \in S$ and define the variable

$$s = \sum_{i \in S \setminus \{k\}} x_i - \sum_{i \in S \setminus \{k\}} D_{i\ell} z_i - y_\ell.$$

Then, the inequalities $s + x_k - D_{k\ell} z_k \le 0$ and $s \le 0$ are both (ℓ, S) inequalities, defined by S and $S \backslash \{k\}$, respectively. We can thus use these two inequalities to define a set $S_{\le}$ and Theorem 3 to derive a new tilted version of the (ℓ, S) inequalities.

Theorem 7 *Let $\ell \in [n]$, $S \subseteq \{1, \dots, \ell\}$ and $F \subseteq S$ be such that $D_{i\ell} < u_i$ for all $i \in F$. For $i \in F$, let $(\bar{\lambda}_i^x, \bar{\lambda}_i^z, \bar{\lambda}_i^t)$ be the solution to (8) with $(a_1, b_2, a_2, b_2, \ell, u) = (1, -D_{i\ell}, 0, 0, 0, u_i)$. Then, the* tilted (ℓ, S) inequality*:*

$$\sum_{i \in S \backslash F} x_i - \sum_{i \in S \backslash F} D_{i\ell} z_i + \sum_{i \in F} (\bar{\lambda}_i^x x_i + \bar{\lambda}_i^t t_i) - y_\ell \le 0, \tag{21}$$

is valid for X^{CLS}.

Proof The proof is by induction on the size of F. When $F = \emptyset$ the inequality (21) is an (ℓ, S) inequality and hence valid for any $\ell \in [n]$ and $S \subseteq \{1, \dots, \ell\}$. Now, suppose that inequality (21) is valid for any $\ell \in [n]$, $S \subseteq \{1, \dots, \ell\}$, and $F \subseteq S$ with $D_{i\ell} < u_i$ for all $i \in F$ and $|F| = c \ge 0$. Now, let $\ell \in [n]$, $\bar{S} \subseteq \{1, \dots, \ell\}$, and $\bar{F} \subseteq \bar{S}$ with $|\bar{F}| = c + 1$. Let $k \in \bar{F}$ and define

$$s = \sum_{i \in \bar{S} \backslash \bar{F}} x_i - \sum_{i \in \bar{S} \backslash \bar{F}} D_{i\ell} z_i + \sum_{i \in \bar{F} \backslash \{k\}} (\bar{\lambda}_i^x x_i + \bar{\lambda}_i^t t_i) - y_\ell.$$

By the induction hypothesis, the inequalities $s + x_k - D_{k\ell} z_k \le 0$ (based on $S = \bar{S}$ and $F = \bar{F} \backslash \{k\}$)) and $s \le 0$ (based on $S = \bar{S} \backslash \{k\}$ and $F = \bar{F} \backslash \{k\}$) are valid for X^{CLS}. We then apply Theorem 3 with $a_1 = 1$, $b_1 = -D_{k\ell}$, $a_2 = 0$, $b_2 = 0$, and obtain $m = D_{k\ell}$ and $\bar{\lambda}_k^x = 0$, and thus the inequality

$$s + \bar{\lambda}_k^x x_k + \bar{\lambda}_k^t t_k \le 0$$

is valid for X^{CLS}, which is equivalent to (21) with $S = \bar{S}$ and $F = \bar{F}$, and hence the induction is complete. □

To separate the tilted (ℓ, S) inequalities, observe that in (21), for each $i \in [n]$, if $i \notin S$, then none of the variables (x_i, z_i, t_i) participate in the inequality; if $i \in S \backslash F$, then these variables contribute the terms $x_i - D_{i\ell} z_i$; and if $i \in F$, then these variables contribute the terms $\bar{\lambda}_i^x \hat{x}_i + \bar{\lambda}_i^t \hat{t}_i$. Thus, given a relaxation solution $(\hat{x}, \hat{y}, \hat{z}, \hat{t})$, a most violated tilted (ℓ, S) inequality can be found efficiently by enumerating each $\ell \in [n]$, and for each fixed ℓ, setting

$$S_\ell := \left\{ i \in \{1, \dots, \ell\} : \max\{\hat{x}_i - D_{i\ell} \hat{z}_i, \bar{\lambda}_i^x \hat{x}_i + \bar{\lambda}_i^t \hat{t}_i\} > 0 \right\},$$
$$F_\ell := \left\{ i \in \{1, \dots, \ell\} : \bar{\lambda}_i^x \hat{x}_i + \bar{\lambda}_i^t \hat{t}_i > \hat{x}_i - D_{i\ell} \hat{z}_i \right\}.$$

The most violated inequality is then obtained by considering the maximum violation among each of these candidate n inequalities. This simple separation procedure for the tilted (ℓ, S) inequalities is similar in complexity to the separation procedure for the (ℓ, S) inequalities described in [3].

Example: Consider an instance of the set X^{CLS} with $n = 3$, $d = (2, 2, 6)$, $u = (10, 8, 6)$ (so $u_i = D_{in}$, $i = 1, 2, 3$), and $f_i(x) = 20x - x^2$, $i = 1, 2, 3$. The (ℓ, S) inequality defined by $\ell = 2$ and $S = \{1, 2\}$ is:

$$(x_1 - 7z_1) + (x_2 - 2z_2) - y_2 \leq 0.$$

Applying the tilting process, we find that with $a_1 = 1, b_1 = D_{i2}, a_2 = 0, b_2 = 0$ and the given functions $f_i(x)$, the tilting coefficients are $\bar{\lambda}_i^x = (20 - D_{i2})/u_i$ and $\bar{\lambda}_i^t = -1/u_i$. Thus, we obtain the following three additional valid inequalities with these fixed choices of ℓ and S:

$$\begin{aligned} (x_1 - 7z_1) + \left(\frac{18}{8}x_2 - \frac{1}{8}t_2\right) - y_2 &\leq 0, \\ \left(\frac{16}{10}x_1 - \frac{1}{10}t_1\right) + (x_2 - 2z_2) - y_2 &\leq 0, \\ \left(\frac{16}{10}x_1 - \frac{1}{10}t_1\right) + \left(\frac{18}{8}x_2 - \frac{1}{8}t_2\right) - y_2 &\leq 0. \end{aligned} \tag{22}$$

The point $\hat{x} = (6, 1, 3)$, $\hat{y} = (4, 3, 0)$, $\hat{z} = (1, 1/2, 1/2)$, $\hat{t} = (60, 12, 42)$ satisfies all the (ℓ, S) inequalities (because it is the convex combination of two feasible solutions to X^{LS}) and also satisfies the secant inequalities, $t_i \geq (f(u_i)/u_i)x_i$, $i = 1, 2, 3$. However, substituting this point into the left-hand-side of (22) yields $(96 - 60)/10 + (18 - 12)/8 - 3 > 0$, and so this point is cut off by a tilted (ℓ, S) inequality.

6 Computational results

In this section, we demonstrate the effectiveness of the TSGFCI on two classes of problems—transportation and lot-sizing problems with concave and fixed costs.

The transportation problem experiments were performed on a heterogeneous set of servers. However, the methods being compared for each given instance were all run on the same machine, so the relative computational time is still a meaningful statistic. The lot-sizing problems were performed on a single Intel Xeon X5650 (24 cores @ 2.66Ghz) server with 128GB of RAM. We used CPLEX 12.6.0 and BARON 14.4.0. Each algorithm was limited to a single thread, a time limit of an hour, and a tolerance of 10^{-6} for relative optimality gap. All problem instances used can be obtained at http://pages.cs.wisc.edu/~conghan/concave/.

6.1 Concave fixed charge transportation problem

We first consider a transportation problem in which flows incur a fixed cost plus concave variable cost. Given a set of facilities I with capacities b_i, $i \in I$ and a set of customers J with demands d_j, $j \in J$ the *Concave Fixed Charge Transportation Problem* (CFCTP) is the optimization problem:

$$\begin{aligned}
\min_{x,z,t}\ & \sum_{i\in I}\sum_{j\in J}\left(t_{ij}+p_{ij}z_{ij}\right) && \text{(CFCTP)}\\
\text{s.t.}\ & \sum_{i\in I} x_{ij}=d_j && \text{for } j\in J\\
& \sum_{j\in J} x_{ij}\le b_i && \text{for } i\in I\\
& t_{ij}\ge f_{ij}(x_{ij}) && \text{for } i\in I,\ j\in J\\
& 0\le x_{ij}\le u_{ij}z_{ij} && \text{for } i\in I,\ j\in J.
\end{aligned}$$

The objective function models both a fixed charge p_{ij} associated with opening arc (i, j) and a cost that is a concave function $f_{ij}(\cdot)$ of the flow on the arc.

We test our methods on randomly generated instances of the CFCTP. Given the sizes of the sets I and J, the remaining data are generated in the following manner. Suppliers and customers are placed uniformly at random in a unit square. The concave functions f_{ij} are given by $f_{ij}(x_{ij}) = w_{ij}x_{ij} - q_{ij}x_{ij}^2$, where w_{ij} is 500 times the Euclidean distance between i and j, and $q_{ij} = w_{ij}/2u_{ij}$, which ensures that f_{ij} is increasing in the range $[0, u_{ij}]$. The fixed costs p_{ij} are drawn uniformly from $[w_{ij}, 5w_{ij}]$. The demand of each customer is drawn uniformly as an integer from [5, 35], and the capacities for each supplier is drawn uniformly from [10, 160] then scaled and rounded to the nearest integer such that the total capacity of the suppliers is 1.3 times the total demand. The capacities on the supplier-customer connections are initially set to $\min\{b_i, d_j\}$ for connection (i, j), and then reduced by repeatedly picking a connection uniformly at random and lowering its capacity by between 1 and 5 units. Once reducing a capacity makes the problem infeasible, the process was stopped. For a fixed problem size, we created a family of ten instances.

The fixed charge network structure of (CFCTP) yields Single Node Flow Set relaxations (10) from which valid "base inequalities" of the form (14) may be generated. To obtain base inequalities that can be tilted, each instance is solved with CPLEX v12.6.0. CPLEX allows the solution of optimization problems with a nonconvex quadratic objective function, so (CFCTP) was reformulated into the equivalent formulation without the t_{ij} variables. The only types of cuts we allowed CPLEX to use were flow cover cuts, and the cuts that were applied by CPLEX at the end of the root node processing were extracted. These cuts were used as the basis of a tilting procedure similar to the one described in Sect. 4.1. The derivation there starts with a valid inequality of the form

$$\sum_{i\in M^+}(\alpha_i x_i+\beta_i z_i)+\sum_{i\in N\setminus M^+}(\pi_i^x x_i+\pi_i^z z_i)\le\gamma$$

where M^+ contains the indices for which the base inequality has $\alpha_i > 0$, $\beta_i < 0$, and $\alpha_i u_i + \beta_i > 0$. The derivation then shows that, for any $k \in M^+$, the inequality $s_k \le \gamma$ is valid, where $s_k := \sum_{i\in M^+\setminus\{k\}}(\alpha_i x_i+\beta_i z_i)+\sum_{i\in N\setminus M^+}(\pi_i^x x_i+\pi_i^z z_i)$, and this assures the tilted inequalities of the form (15) are valid under the assumptions of Theorem 4. In Sect. 4.1, validity of $s_k \le \gamma$ is checked by using the particular structure of the set $X^{\mathtt{SNFS}}$. Since we cannot be sure that the cuts generated by CPLEX are valid for such a set, in our implementation we computationally verify that the inequalities $s_k \leq \gamma$ are valid over the feasible region, for each $k \in M^+$. We do this by computing max s_k over the feasible region of the mixed-integer *linear* set of the instance, and then check that the optimal value is not greater than γ. Because these are relatively small mixed-integer linear programs, it took less than a second for each problem instance to verify validity of all possible $s_k \leq \gamma$ constraints across all base cuts generated by CPLEX.

CPLEX also is often able to tighten the upper bounds u_{ij} on the flow variables, and we also extract and use these improved values in computing the lifting coefficient via (8). In our experiments, we consider tilting only the variables in the set M^+ (via Theorem 4) and do not test the more general TSGFCI of Theorem 6.

Using the extracted base valid inequalities from CPLEX, tilted inequalities (15) are added in a separation loop using the optimal lifting set $F = F^*$ (defined in (16)) until no more inequalities can be separated. In addition, for each base inequality we also add all tilted inequalities (15) with $|F| \leq 2$. We remark that this implementation potentially underestimates the benefits from using the tilted inequalities, since we are using valid inequalities that are generated without knowledge of the tilting procedure. An integrated procedure that directly searches for tilted inequalities could potentially identify additional violated inequalities.

We compare the performance of four different solution approaches: using CPLEX with default options on the reformulation that eliminates the t variables, using BARON on the original problem (B), using BARON on the problem supplemented with implied bounds and valid inequalities extracted by CPLEX (BVI), and finally using BARON with the settings of (BVI) plus the additional tilted inequalities (15) added (BT). We could not test the performance of CPLEX with the inequalities (15) because CPLEX does not allow nonconvex constraints, and so the formulation used in CPLEX does not contain the necessary t variables.

We present a summary of our computational results in Tables 1 and 2 and Fig. 2. We omit the results for CPLEX in the tables since the methods using BARON significantly outperform it. The numbers reported for each problem family in the tables are averaged over the 10 instances.

Table 1 reports statistics on the cuts and initial relaxations. The *Number of Cuts* columns denote the average number of each class of cuts generated over the 10 instances. The *Initial Gap* columns indicate the the gap between the solution of the LP relaxation of the problem and the best feasible solution among all methods. The SEC, VI, and T columns correspond to the secant relaxation, the secant relaxation with valid inequalities from CPLEX, and the secant relaxation with valid inequalities from CPLEX and tilted inequalities (15), respectively. The *Root Gap* columns indicate the gap between the lower bound at the root node of BARON and the best solution among all methods, which includes the effect of cuts, bounds tightening, and other preprocessing that BARON performs at the root node. For all these instances, the time taken to generate the cuts is less than a few seconds. We find that the processing that BARON performs at the root node significantly improves the relaxation gaps. The initial gap is often significantly improved by adding the CPLEX valid inequalities. However, the bounds obtained after BARON processes the root node are not affected by the addition of these valid inequalities to the formulation. On the other hand, adding the tilted inequalities improves the initial gap modestly, but leads to significantly reduced

Table 1 Average number of cuts and relaxation gaps for the CFCTP

Instance family		Number of cuts		Initial gap			Root gap		
Suppliers	Customers	Flow covers	Tilts	SEC (%)	VI (%)	T (%)	B (%)	BVI (%)	BT (%)
10	10	36.1	41.9	16.10	8.68	6.35	7.85	7.85	4.10
10	15	43.9	45.9	15.07	8.06	5.63	7.61	7.61	3.50
15	12	45.1	61	14.80	8.46	6.86	7.10	7.10	4.02
12	18	54.6	77.9	18.03	10.47	8.17	8.41	8.41	4.43
15	15	50.5	59.1	14.53	8.71	6.89	7.72	7.72	4.42
18	18	67.1	102.7	15.23	8.62	6.79	6.94	6.94	3.67

Table 2 Number of instances solved and average ending optimality gaps for the transportation problem

Instance family		Instances solved			Final gap		
Suppliers	Customers	B	BVI	BT	B (%)	BVI (%)	BT (%)
10	10	10	10	10	0.00	0.00	0.00
10	15	3	3	7	3.19	2.27	0.95
15	12	4	4	8	2.16	2.04	0.40
12	18	0	0	3	5.96	5.45	2.52
15	15	1	1	5	4.79	4.61	2.09
18	18	0	0	0	5.47	5.36	2.59

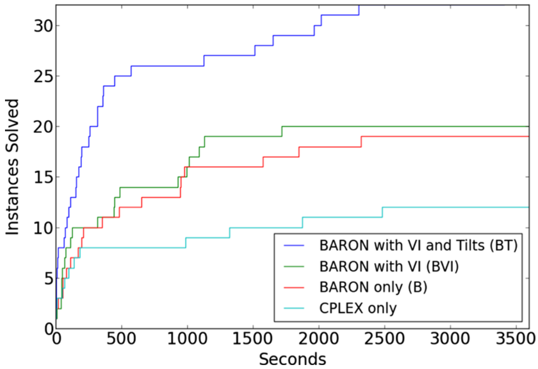

Fig. 2 Cumulative distribution plots of solution times for the transportation problem. The plots from top to bottom correspond to the order of the labels in the legend

gap after BARON processes the root node, closing nearly 50% of the gap relative to BARON's root node relaxation without the tilted inequalities.

Table 2 reports results obtained after running the methods to termination or the time limit. *Instances Solved* indicates how many out of 10 instances each method is able to solve within an hour. The *Final Gap* columns indicates the average gap between the lower bound the method achieves within the time limit and the best solution among all methods. We observe that BT solves almost twice as many instances within the time limit (32 versus 18/19 for B and BVI), and for unsolved instances BT is able to close substantially more of the gap than BVI. The set of instances solved for BT is a strict superset of those for BVI, which is in turn a superset of those solved for B (and CPLEX).

Figure 2 presents a plot of the cumulative distribution of solution times over the 32 instances that are solved by at least one method. Each plot indicates the number of instances that have been solved by a certain time. The performance for BT significantly dominates all other methods, and BVI shows a slight improvement over B, demonstrating that the valid inequalities from CPLEX make a small but significant difference, whereas the tilted valid inequalities yield large improvements in computation time.

6.2 Lot-sizing problem with concave costs

We now consider an adaptation of the canonical *Lot-Sizing Problem* with concave costs. Using the notation from Sect. 5, the concave-cost lot-sizing problem is:

$$\min_{x,y,z,t} \left\{ \sum_{i\in[n]} (h_i y_i + p_i z_i + t_i) : (x, y, z, t) \in X^{\mathrm{CLS}} \right\},$$

where, for each $i \in [n]$, the variables t_i encode the variable production cost, $f_i(x_i)$, h_i represents the per-unit cost of holding an item in inventory, and p_i represents the fixed setup cost that is incurred if $x_i > 0$.

We created our test instances based on an extension of the instance generation process used in [1]. Each family of instances is identified by three parameters — the number of time periods $n \in \{70, 90\}$, the ratio between the production capacity and total demand $c \in \{3, 10, \text{uncap}\}$, and the approximate setup cost $r \in \{200, 500\}$. When $c \in \{3, 10\}$ the production capacity in period $i \in [n]$, u_i, is an integer drawn uniformly from $[0.75c\bar{d}, 1.25c\bar{d}]$, and when $c = \text{uncap}$, the production is uncapacitated, so we set $u_i = D_{in}$. There exist polynomial algorithms for solving the concave-cost lot sizing problem in the uncapacitated case (e.g., [10,14,15,27]). However, we are still interested in conducting experiments on this case to investigate the impact of our cuts when a general purpose solution approach is applied to this problem. For each period $i \in [n]$, per unit holding cost h_i is 10, and the demand d_i is an integer drawn uniformly from [1, 19]. Let $\bar{d}$ denote the average demand of an instance. The setup cost p_i is an integer drawn uniformly from $[9r, 1.1r]$. For $i \in [n]$, we use $f_i(x_i) = w_i x_i - q_i x_i^2$, where w_i is an integer drawn uniformly from [81, 119] and $q_i = w_i/2u_i$, which ensures that f_i is nondecreasing on $[0, u_i]$. We created five instances for each combination of the parameters n, c, r, and the results reported in all tables are averages over these five instances.

We use (ℓ, S) inequalities and tilted (ℓ, S) inequalities in a cut-and-branch fashion. We first build and solve the *secant relaxation*, in which the binary variables are relaxed to be continuous, and the constraints $t_i \geq f(x_i)$ are replaced by $t_i \geq (f(u_i)/u_i)x_i$.

Table 3 Average relaxation gaps for the lot-sizing problem

Instance family			Initial gap			Root gap		
n	c	r	SEC (%)	LS (%)	T (%)	B (%)	BLS (%)	BT (%)
70	3	200	10.18	6.25	6.25	0.94	1.08	1.12
70	3	500	9.61	7.42	7.42	0.39	0.51	0.49
70	10	200	31.22	9.36	4.40	6.48	6.48	0.20
70	10	500	27.06	7.82	7.07	2.13	2.13	0.50
70	Uncap	200	53.06	25.29	2.89	23.25	23.10	0.04
70	Uncap	500	53.95	20.53	4.84	16.28	10.94	0.04
90	3	200	10.53	6.08	6.08	1.34	1.53	1.54
90	3	500	7.44	5.54	5.54	1.06	1.13	1.16
90	10	200	30.96	8.31	3.41	6.13	6.13	0.12
90	10	500	27.72	7.29	6.61	2.28	2.28	0.64
90	Uncap	200	56.15	26.11	2.39	24.39	24.28	0.01
90	Uncap	500	57.30	20.72	3.80	17.80	14.61	0.01

Given the solution of this relaxation, the most violated inequality in the class being used ((ℓ, S) inequalities or tilted (ℓ, S) inequalities) is found and added to the relaxation. The relaxation is then re-solved and the process repeated until no further violated cuts are identified. Note that (ℓ, S) inequalities are a special case of tilted (ℓ, S) inequalities, so when using tilted (ℓ, S) inequalities there is no need to search explicitly for (ℓ, S) inequalities. After this process terminates, the cuts identified throughout this process are included in the formulation that is passed to BARON.

We compare the performance of four different solution approaches: using CPLEX with default options on the reformulation that eliminates the t variables; using BARON to solve the original formulation (B); using BARON to solve the formulation supplemented with (ℓ, S) inequalities (BLS); and finally using BARON to solve the formulation supplemented with tilted (ℓ, S) inequalities (BT).

We first investigate the impact of the tilted (ℓ, S) inequalities on the initial relaxation. For a given relaxation and instance, we compute the optimality gap as $(z^* - LB)/z^*$ where z^* is the optimal value of that instance and LB is the lower bound produced by that relaxation. Table 3 presents the optimality gaps obtained before and after BARON has processed the root node. The *Initial Gap* columns report the average optimality gaps of the secant relaxation (SEC), the secant relaxation with (ℓ, S) inequalities (LS), and the secant relaxation with tilted (ℓ, S) inequalities (T). The *Root Gap* columns indicate the gap obtained by BARON after processing the root node, e.g., by adding general-purpose valid inequalities performing bounds tightening, and other preprocessing techniques. Considering first the initial gaps, we find that using the (ℓ, S) inequalities yields significantly smaller gaps than the basic secant relaxation, and that tilted (ℓ, S) inequalities further reduce the gaps significantly when $c = 10$ or $c =$ uncap, but have little impact when $c = 3$. We also find the impact from the tilted (ℓ, S) inequalities to be relatively smaller when the fixed cost parameter, r, is higher.

Table 4 Average number of cuts and time to generate them for the lot-sizing problem

Instance family			Number of cuts		Time (s)	
n	c	r	(ℓ, S)	Tilt (ℓ, S)	(ℓ, S)	Tilt (ℓ, S)
70	3	200	36.8	39.4	0.9	0.9
70	3	500	24.4	29.8	0.6	0.7
70	10	200	127.8	183.2	2.9	4.3
70	10	500	132.0	154.2	3.0	3.5
70	Uncap	200	794.2	1269.6	19.7	42.6
70	Uncap	500	626.0	979.2	16.0	32.9
90	3	200	53.8	54.4	1.7	1.7
90	3	500	32.2	36.2	1.0	1.1
90	10	200	155.6	212.6	4.8	6.7
90	10	500	164.8	181.6	5.0	5.6
90	Uncap	200	1558.2	2442.2	58.4	149.7
90	Uncap	500	1259.6	1957.8	47.8	116.2

When c is relatively small, or r is relatively high, the relaxation solutions will tend to have x_i near its upper bound u_i whenever $z_i = 1$, and hence the error between the secant relaxation and the true concave cost is expected to be lower, which explains why the tilted (ℓ, S) inequalities have less relatively impact in those cases. We also find that in all cases BARON is able to reduce the root gaps significantly, but that the relative performance of using (ℓ, S) inequalities and tilted (ℓ, S) inequalities is unchanged.

In Table 4 we present information on the number of cuts generated when using (ℓ, S) inequalities and tilted (ℓ, S) inequalities, and the time spent performing the cut-generation process (including separation time and time re-solving the relaxation). In general, the number of cuts added increases as n or c increase or when r decreases.

Finally, Table 5 summarizes the results for solving the instances to optimality, and Fig. 3 presents cumulative distribution plots of the solution times for the different methods. The *Instances Solved* columns present how many of the five instances for the given set of parameters were solved, and the *Final Gap* columns present the average gap between the lower bound the method achieves within the time limit and the best solution among all methods. From Table 5 we find that when $c = 10$ or $c =$ uncap, the tilted (ℓ, S) inequalities enable solving many more instances to optimality within the time limit, and that without them large optimality gaps may remain. On the other hand, when $c = 3$, the results from using tilted (ℓ, S) inequalities are similar to those obtained without them, which is consistent with the observations on the effect of the tilted (ℓ, S) inequalities on the root gaps. Figure 3 demonstrates that using the tilted (ℓ, S) inequalities can significantly reduce solution times. For example, more than half the instances can be solved in under 250 s when using the tilted (ℓ, S) inequalities, whereas less than half of the instances are solved within the 1 h time limit when the tilted (ℓ, S) inequalities are not used.

Table 5 Number of instances solved and average ending optimality gaps for the lot-sizing problem

Instance family			Instances solved			Final gap		
n	c	r	B	BLS	BT	B (%)	BLS (%)	BT (%)
70	3	200	4	4	5	0.02	0.03	0.00
70	3	500	5	5	5	0.00	0.00	0.00
70	10	200	0	0	5	5.82	3.04	0.00
70	10	500	5	5	5	0.00	0.00	0.00
70	Uncap	200	0	0	5	17.39	17.44	0.00
70	Uncap	500	0	2	5	10.29	2.35	0.00
90	3	200	1	1	1	0.31	0.34	0.33
90	3	500	3	4	2	0.02	0.06	0.11
90	10	200	0	0	5	5.51	3.35	0.00
90	10	500	2	3	5	0.55	0.44	0.00
90	Uncap	200	0	0	5	21.04	17.39	0.00
90	Uncap	500	0	0	5	12.51	9.77	0.00

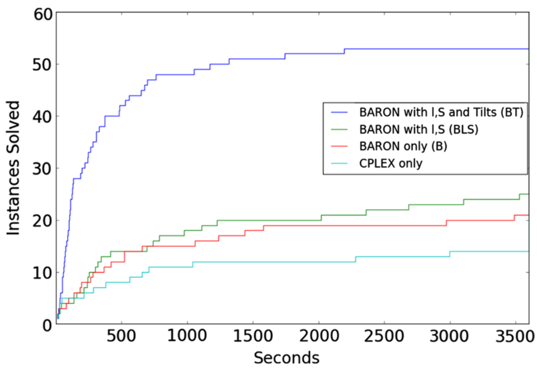

Fig. 3 Cumulative distribution plots of solution times for the lot-sizing problem. The plots from top to bottom correspond to the order of the labels in the legend

7 Conclusion

We study valid inequalities for a mixed-integer nonlinear set having binary indicator variables and separable concave constraints. We introduce a technique that obtains valid inequalities for this set by applying a tilting procedure to inequalities that are known for the set ignoring the concave constraints. We apply this procedure to versions of this set in which the linear constraints correspond to network flow and lot-sizing problems, and find that the new inequalities yield significant reductions in solution times. In future work, it will be interesting to test the proposed procedure on additional problems having this network structure, and to investigate the application of the proposed tilting procedure to other mixed-integer nonlinear structures.

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Appendix: Proof of Theorem 5

Before stating the proof, which roughly follows the structure of the proof of Theorem 6 in [26], we establish some notation and some preliminary results. We denote a point $p \in X_f^{\mathtt{CSNFS}}$ as

$$p = (({}^1x, {}^1z, {}^1t), ({}^2x, {}^2z, {}^2t), ({}^3x, {}^3z, {}^3t), ({}^4x, {}^4z, {}^4t), ({}^5x, {}^5z, {}^5t))$$

where the left superscript indicates if the coordinates correspond to the values (1) $C^+ \backslash F$, (2) F, (3) $N^+ \backslash C^+$, (4) L^-, or (5) $N^- \backslash L^-$. Let ${}^j u$ denote the vector of u_k values associated with the jth set defined above.

We prove the theorem by giving $3n$ affinely independent points of $X_f^{\mathtt{CSNFS}}$ that lie on the face defined by (18) and define the coefficients in the inequality (18) up to a scalar multiple. Since the solution with all variables set to zero is in $X_f^{\mathtt{CSNFS}}$, but does not lie on the face defined by (18), this implies that $X_f^{\mathtt{CSNFS}}$ is full-dimensional.

We tackle the proof in four separate parts. We first describe $3|C^+ \backslash F|$ points, then $3|F|$ points, followed by $3|N^+ \backslash C^+|$ points, and finally the remaining $3|N^-|$ points. The points are defined in the arguments that follow. We let

$$\sum_{k \in N} (\pi_k^x x_k + \pi_k^z z_k + \pi_k^t t_k) = \pi_0. \tag{23}$$

denote the equality defined by these (yet to be defined) points. In the proofs that follow, we use $f({}^i x)$ to denote the vector of function values consisting of $f_k(x_k)$ for k corresponding to the set indicated by i. It can readily be checked that the $3n$ points we define are in $X_f^{\mathtt{CSNFS}}$ and satisfy (18) at equality.

Lemma 2 *The equality* (23) *has the form*

$$\sigma \left(\sum_{k \in C^+ \backslash F} \left(x_k - (u_k - \mu)^+ z_k \right) \right) + \sum_{k \in N \backslash (C^+ \backslash F)} (\pi_k^x x_k + \pi_k^z z_k + \pi_k^t t_k) = \pi_0.$$

Proof The proof proceeds by showing that π_k^t, π_k^x, and π_k^z (in this order) for $k \in C^+ \backslash F$ have the form given in the statement of the lemma.

We describe a set of $3|C^+ \backslash F|$ points. From here on, we will use $\overline{1}$ to denote the all-ones vector. We first define points for $k \in C^+ \backslash F$ where $u_k \geq \mu$:

$$\begin{aligned}
a^k &= (({}^1u - \mu e_k, \overline{1}, f({}^1u - \mu e_k)), ({}^2u, \overline{1}, f({}^2u)), 0, \ldots, 0) \\
b^k &= (({}^1u - u_k e_k, \overline{1} - e_k, f({}^1u - u_k e_k)), ({}^2u, \overline{1}, f({}^2u)), 0, \ldots, 0) \\
c^k &= (({}^1u - u_k e_k, \overline{1} - e_k, f({}^1u - u_k e_k) + 1), ({}^2u, \overline{1}, f({}^2u)), 0, \ldots, 0).
\end{aligned}$$

When $u_k < \mu$, we pick a scaling term

$$\eta_k = \frac{\sum_{j \in C^+ \setminus F} u_j - \mu}{\sum_{j \in C^+ \setminus F} u_j - u_k}.$$

By assumption (ii), it is straightforward to see that $0 < \eta_k < 1$ and

$$\eta_k \Big(\sum_{j \in C^+ \setminus (F \cup \{k\})} u_j \Big) + \sum_{j \in F} u_j = d.$$

Then, we form the following points for $k \in C^+ \setminus F$ with $u_k < \mu$:

$$\begin{aligned}
a^k &= ((\eta_k({}^1u - u_k e_k), \overline{1}, f(\eta_k({}^1u - u_k e_k))), ({}^2u, \overline{1}, f({}^2u)), 0, \dots, 0) \\
b^k &= ((\eta_k({}^1u - u_k e_k), \overline{1} - e_k, f(\eta_k({}^1u - u_k e_k))), ({}^2u, \overline{1}, f({}^2u)), 0, \dots, 0) \\
c^k &= ((\eta_k({}^1u - u_k e_k), \overline{1} - e_k, f(\eta_k({}^1u - u_k e_k)) + \overline{1}), ({}^2u, \overline{1}, f({}^2u)), 0, \dots, 0).
\end{aligned}$$

By comparing b^k and c^k for each $k \in C^+ \setminus F$, we get $\pi_k^t = 0$ for $k \in C^+ \setminus F$.

For the points a^k corresponding to $u_k \geq \mu$ and a^k, b^k, c^k to $u_k < \mu$, the sum of the terms in C^+ for these points gives us d and so

$$\sum_{j \in C^+ \setminus F} x_j = d - \sum_{j \in F} u_j. \tag{24}$$

Now consider the a^k points for $k \in C^+ \setminus F$. The $|C^+ \setminus F|$ vectors in the two sets

$$\begin{aligned}
\Big\{ {}^1u - \mu e_k \ : \ k \in C^+ \setminus F, u_k \geq \mu \Big\} &\subseteq \mathbb{R}^{|C^+ \setminus F|} \\
\Big\{ \eta_k({}^1u - u e_k) \ : \ k \in C^+ \setminus F, u_k < \mu \Big\} &\subseteq \mathbb{R}^{|C^+ \setminus F|}
\end{aligned}$$

are linearly independent and have the same sum. This implies that for all the $k \in C^+ \setminus F$ we have $\pi_k^x = \sigma$ for some $\sigma \in \mathbb{R}$. We obtain the equality

$$\begin{aligned}
\pi_0 &= \sum_{j \in C^+ \setminus F} \sigma x_j + \sum_{j \in C^+ \setminus F} \pi_j^z + \sum_{j \in F} \big(\pi_j^x x_j + \pi_j^z z_j + \pi_j^t t_j \big) \\
&= \sigma \Big(d - \sum_{j \in F} u_j \Big) + \sum_{j \in C^+ \setminus F} \pi_j^z + \sum_{j \in F} \big(\pi_j^x u_j + \pi_j^z + \pi_j^t f_j(u_j) \big)
\end{aligned} \tag{25}$$

where the second equality follows from (24).

For each b_k for $k \in C^+ \setminus F$ where $u_k \geq \mu$, we get

$$\sigma \Big(\sum_{j \in C^+ \setminus (F \cup \{k\})} u_j \Big) + \sum_{j \in C^+ \setminus (F \cup \{k\})} \pi_j^z + \sum_{j \in F} \big(\pi_j^x u_j + \pi_j^z + \pi_j^t f_j(u_j) \big) = \pi_0. \tag{26}$$

Subtracting (26) from (25), we obtain

$$\sigma\Big(d - \sum_{j\in C^+\setminus\{k\}} u_j\Big) + \pi_k^z = 0,$$

which reduces to $\sigma(\mu - u_k) = \pi_k^z$. As for $k \in C^+\setminus F$ where $u_k < \mu$, the b^k points gives us

$$\begin{aligned}\pi_0 &= \sum_{j\in C^+\setminus F} \sigma x_j + \sum_{j\in C^+\setminus(F\cup\{k\})} \pi_j^z + \sum_{j\in F}(\pi_j^x x_j + \pi_j^z z_j + \pi_j^t t_j)\\ &= \sigma\Big(d - \sum_{j\in F} u_j\Big) + \sum_{j\in C^+\setminus(F\cup\{k\})} \pi_j^z + \sum_{j\in F}(\pi_j^x u_j + \pi_j^z + \pi_j^t f_j(u_j)) \qquad (27)\end{aligned}$$

using (24) again. By subtracting (27) from (25), we have $\pi_k^z = 0$. □

Lemma 3 *The coefficients $\pi_k^x, \pi_k^z, \pi_k^t$ for equality (23) when $k \in F$ satisfy the following: $\pi_k^x = \sigma\bar{\lambda}_k^x, \pi_k^z = \sigma\bar{\lambda}_k^z$, and $\pi_k^t = \sigma\bar{\lambda}_k^t$, where $\bar{\lambda}_k^x, \bar{\lambda}_k^z, \bar{\lambda}_k^t$ are the terms obtained by applying Theorem 3 for the case where $a_1 = 1, b_1 = \mu - u_k$, and $a_2 = b_2 = 0$.*

Proof Recall from assumption (v) that $u_k > \mu$ for $k \in F$. We define the following points associated with each $k \in F$:

$$\begin{aligned}a^k &= ((^1u, \bar{1}, f(^1u)), (^2u - \mu e_k, \bar{1}, f(^2u - \mu e_k)), 0, \dots, 0)\\ b^k &= ((^1u, \bar{1}, f(^1u)), (^2u - u_k e_k, \bar{1}, f(^2u - u_k e_k)), 0, \dots, 0)\\ c^k &= ((^1u, \bar{1}, f(^1u)), (^2u - u_k e_k, \bar{1} - e_k, f(^2u - u_k e_k)), 0, \dots, 0).\end{aligned}$$

Comparing b^k and c^k for $k \in F$, we get that $\pi_k^z = 0$ for $k \in F$, which is the same as the value of $\bar{\lambda}_k^z$ derived from (8) since $\ell_k = f(\ell_k) = 0$.

From a^k and b^k, we know that

$$\pi_k^x(u_k - \mu) + \pi_k^t f(u_k - \mu) = 0. \qquad (28)$$

Furthermore, from comparing a^k and a^l for some $l \in C^+\setminus F$ where $u_l \geq \mu$, we have

$$\begin{aligned}0 &= \sigma\mu - \pi_k^x\mu - \pi_k^t\big(f_k(u_k) - f_k(u_k - \mu)\big)\\ &= \sigma\mu + \pi_k^x(u_k - \mu) + \pi_k^x f_k(u_k - \mu) - \pi_k^t f_k(u_k) - \pi_k^x u_k\\ &= \sigma\mu - \pi_k^t f_k(u_k) - \pi_k^x u_k \qquad (29)\end{aligned}$$

where (29) is obtained by applying (28). Equations (29) and (28) allow us to form the following system of linear equations akin to (8) from Theorem 3:

$$\begin{pmatrix} 0 & 1 & 0\\ u_k - \mu & 1 & f_k(u_k - \mu)\\ u_k & 1 & f_k(u_k)\end{pmatrix}\cdot\begin{pmatrix}\pi_k^x\\ 0\\ \pi_k^t\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \sigma\mu\end{pmatrix}$$

which gives us $\pi_k^x = \sigma\bar{\lambda}_k^x, \pi_k^t = \sigma\bar{\lambda}_k^t$, where $\bar{\lambda}_k^x, \bar{\lambda}_k^t$ are the terms obtained from (8) for the case where $a_1 = 1, b_1 = \mu - u_k, a_2 = 0, b_2 = 0$. □

Lemma 4 $\pi_0 = \sigma d(C^+, \emptyset)$.

Proof We begin with (25) and consider the coefficients we obtained in Lemmas 2 and 3:

$$\begin{aligned}
\pi_0 &= \sigma\Big(d - \sum_{j\in F} u_j\Big) + \sum_{j\in C^+\setminus F} \pi_j^z + \sum_{j\in F}\big(\pi_j^x u_j + \pi_j^z + \pi_j^t f_j(u_j)\big) \\
&= \sigma\Big(d - \sum_{j\in F} u_j\Big) - \sum_{j\in C^+\setminus F} \sigma(u_k-\mu)^+ + \sum_{j\in F}\big(\pi_j^x u_j + \pi_j^z + \pi_j^t f_j(u_j)\big) \\
&= \sigma\Big(d - \sum_{j\in F} u_j\Big) - \sum_{j\in C^+\setminus F} \sigma(u_k-\mu)^+ + \sum_{j\in F} \sigma\mu_k \\
&= \sigma\Big(d - \sum_{j\in C^+} (u_k-\mu)^+\Big).
\end{aligned}$$

The second equality follows from Lemma 2, the third from (29) and Lemma 3, and the final from the fact that all $\mu_k > \mu$ for all $k \in F$. □

Building on the previous lemmas, we now complete the proof of Theorem 5.

Proof of Theorem 5 We first define $3|N^+\setminus C^+|$ more points. From assumption (ii) of the theorem, we know that there is some coordinate $l \in C^+$ where $u_l > \mu$. Depending on whether l is in $C^+\setminus F$ or in F, we define the following vector $v \in \mathbb{R}^{|C^+|}$ accordingly:

$$v^{C^+} = \begin{cases} ((^1u - u_l e_l, \bar{1} - e_l, f(^1u - u_l e_l)), (^2u, \bar{1}, f(^2u))), & \text{if } l \in C^+\setminus F \\ ((^1u, \bar{1}, f(^1u)), (^2u - u_l e_l, \bar{1}, f(^2u - u_l e_l))), & \text{if } l \in F. \end{cases}$$

This allows us to consider the following points:

$$\begin{aligned}
&\text{for } k \in N^+\setminus C^+ : \\
&\quad a^k = (v^{C^+}, (0, e_k, 0), 0, \ldots, 0) \\
&\quad b^k = (v^{C^+}, (\epsilon e_k, e_k, f(\epsilon e_k)), 0, \ldots, 0) \\
&\quad c^k = (v^{C^+}, (\epsilon e_k, e_k, f(\epsilon e_k) + 1), 0, \ldots, 0),
\end{aligned}$$

By comparing a^k, b^k and c^k for each k, we know that $\pi_k^x = \pi_k^t = 0$. Furthermore, by comparing a^k to b^l (as defined in Lemma 2 or 3 depending on which set index l belongs to), we have $\pi_k^z = 0$.

We now focus on points corresponding to N^-. Let

$$v^{N^+} = ((^1u, \bar{1}, f(^1u)), (^2u, \bar{1}, f(^2u)), (0, 0, 0)).$$

We define the following $3|L^-|$ points for $k \in L^-$:

$$a^k = (v^{N^+}, (\mu e_k, e_k, f(\mu e_k)), (0,0,0)),$$
$$b^k = (v^{N^+}, (u_k e_k, e_k, f(u_k e_k)), (0,0,0)),$$
$$c^k = (v^{N^+}, (u_k e_k, e_k, f(u_k e_k) + \overline{1}), (0,0,0)).$$

From comparing a^k, b^k, and c_k we get $\pi_k^x = \pi_k^t = 0$. Now consider the point a^k and compare this to the point a^l again. By subtracting a^l from a^k and applying Lemma 2, we get $\pi_k^z = -\sigma\mu$. Recall from assumption (iii) of Theorem 5 that $u_k > \mu$ for $k \in L^-$, so $\pi_k^z = -\sigma \min(u_k, \mu)$ as desired.

For the last $3|N^- \backslash L^-|$ points, we use the following scaling terms:

$$\eta = \frac{\sum_{j \in C^+ \backslash F} u_j - \mu}{\sum_{j \in C^+ \backslash F} u_j}, \quad \eta^\epsilon = \frac{\sum_{j \in C^+ \backslash F} u_j - (\mu - \epsilon)}{\sum_{j \in C^+ \backslash F} u_j}.$$

For $\epsilon > 0$ small enough, η and η^ϵ are strictly between 0 and 1 due to assumption (vi). We define:

$$v^{N \backslash L^-} = ((\eta\, {}^1u, \overline{1}, f(\eta\, {}^1u)), ({}^2u, \overline{1}, f({}^2u)), (0,0,0), (0,0,0)),$$
$$v^{N \backslash L^-, \epsilon} = ((\eta^\epsilon\, {}^1u, \overline{1}, f(\eta^\epsilon\, {}^1u)), ({}^2u, \overline{1}, f({}^2u)), (0,0,0), (0,0,0)),$$

which we use to define the following points for each $k \in N^- \backslash L^-$:

$$a^k = (v^{N \backslash L^-}, (0, e_k, 0)),$$
$$b^k = (v^{N \backslash L^-, \epsilon}, (\epsilon e_k, e_k, f(\epsilon e_k))),$$
$$c^k = (v^{N \backslash L^-, \epsilon}, (\epsilon e_k, e_k, f(\epsilon e_k) + \overline{1})).$$

We have $\pi_k^t = 0$ from comparing b^k and c^k, and comparing a^k to a^l again shows that $\pi_k^z = 0$. Comparing a^k and b^k and using Lemma 2, this implies that $\pi_k^x = -\sigma$.

Thus, we know that any equality that touches these $3n$ points has to have a form of inequality (18) up to a scaling factor. We know that inequality (18) is valid for our set, so all that remains is to prove that this equality contains a facet instead of the entire set $X_f^{\mathtt{CSNFS}}$. The point given by $((0, \overline{1}, 0), 0, \cdots, 0)$ does not satisfy this equality. □

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